



TESIS DOCTORAL

Essays on Contests and Conflicts

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Essays on Contests and Conflicts

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Abstract

My dissertation consists of three chapters that analyze the behavior of decision makers and their interactions in situations where agents engage in costly effort investment in order to earn a prize such as property rights, natural resources, market shares, etc. This analysis allows competition between economic agents when the property rights are not clearly defined, imperfectly enforced, or absent completely by design or naturally. As property rights are absent in various economic environments, my dissertation allows applications of political economy, litigation, sports, etc.

First chapter is a joint work with my supervisor Luis C. Corchón and it is a published work in *Journal of Economic Behavior and Organization*. Our paper concentrates on the Political Economy implications of contests and analyzes a conflict over a resource between two contestants that differ in effectiveness of employing their investments on forces. We consider, for the cases of complete and asymmetric information, a pre-existent resource distribution and we analyze the effects of that distribution on sustaining peace. We find that under complete information there is always some distribution that achieves peace. If the contestants are similar, the set of such distributions are larger. Thus, peace may be achieved rather simply. Under asymmetric information we show that if the asymmetric information is small there may be no distribution of the resource that sustains peace in equilibrium. Thus, when there is asymmetric information or misinterpretation of the strength of the other contestant, the cheap-talk game of declaring war has consequences that lead to war even if the contestants have the chance of obtaining a part of the resource under conflict without aggression.

Second chapter focuses on the possibility of ties in contests in general. The possibility of a tie may naturally be existent in the contest such as an impasse in a military conflict, the imperfect credibility of the prize granting authority in lobbying. It may also be imposed by design as in sport events such as football, chess, etc and promotional contests where denying promotion too all is a granted right of an employer. My paper introduces a new functional form allowing the possibility of a draw in a contest as a function of the expenses spent by the contestants and analyses the game induced by assigning a non-negative price for the outcome of the tie. I also build a dataset from four major football leagues of Europe including information on market values of the teams, and the result of each match for ten seasons. I use this data as a first assessment of the empirical performance of contest success functions for ties in the literature. According to my functional form, probability of a tie reaches a maximum whenever contestants exert equal amounts of effort regardless of the magnitude of these efforts. It increases when a player with less effort increases his effort, and decreases otherwise. In the unique equilibrium, players spend more for the contest with ties compared to the amount they spend for the contest without ties, even if the prize obtained in case of a tie is equivalent to losing the contest. This result implies that players compete more even if their expected prize is lower than the one when there is no possibility of a tie. Equilibrium also indicates that the equilibrium efforts do not

depend on the prize allocated in case of a tie. Thus, if a contest designer wishes to obtain the largest effort from the contest, she should admit the possibility of a draw and assign zero prize for it. Moreover, if there is a constrained player in terms of resources an increase in the tie prize decreases the total expenditures spent by reducing the incentives for effort of the unconstrained player. The empirical application shows that my contest success function has promising results in determining the likelihood of various possible outcomes of the contest.

Third chapter is also a Political Economy application of a contest and considers a two-player dynamic conflict in which, after an initial stage of arming decisions, players decide whether to stop or continuing fighting at each stage of conflict. The war does not stop until both sides decide to stop. Arms are destructive to rival forces and relative amount of forces determines the destructive power and bargaining power over the resource at each stage. In the subgame that starts after the initial arming decisions, war does not start at all if players discount future heavily or one side has a very large or a very small advantage in forces. Given that war starts, the smaller the relative advantage in forces the longer the conflict lasts. In the subgame perfect equilibrium of this game I find that the unique equilibrium is the one in which armed peace prevails.

Resumen

Mi tesis consiste tres capítulos que analizan el comportamiento de los tomadores de las decisiones y sus interacciones en situaciones donde los agentes se dedican a la inversión costoso esfuerzo con el fin de ganar un premio como los derechos de propiedad, los recursos naturales, cuotas de mercado, etc. Este análisis permite la competencia entre los agentes económicos, cuando los derechos de propiedad no están claramente definidos, de manera imperfecta forzada, o ausente por completo por su diseño o de forma natural. Como los derechos de propiedad están ausentes en diversos entornos económicos, mi tesis permite a las aplicaciones de la economía política, litigios, deportes, etc.

El primer capítulo es un trabajo conjunto con mi supervisor Luis C. Corchón y es una obra publicada en *Journal of Economic Behavior and Organization*. Nuestro trabajo se centra en las implicaciones Economía Política de concursos y analiza un conflicto por un recurso entre dos concursantes que difieren en la eficacia del empleo de sus inversiones en las fuerzas. Consideramos que, para los casos de información completa y asimétrica, una distribución de los recursos pre-existentes y se analizan los efectos de que la distribución en el mantenimiento de la paz. Nos encontramos con que la sección de información completa, siempre hay alguna distribución que logra la paz. Si los concursantes son similares, el conjunto de estas distribuciones son más grandes. Por lo tanto, la paz se puede lograr simplemente. Según la información asimétrica se demuestra que si la información asimétrica es pequeña puede que no haya distribución del recurso que sustenta la paz en el equilibrio. Por lo tanto, cuando hay información asimétrica o mala interpretación de la fuerza del otro competidor, el juego con parloteo de declarar guerra tiene consecuencias que conducir a la guerra, aunque los concursantes tienen la oportunidad de obtener una parte de los recursos situados bajo conflicto sin agresión.

El segundo capítulo se centra en la posibilidad de empates en los concursos en general. La posibilidad de un empate se puede existir naturalmente en el concurso como un impasse en un conflicto militar, la credibilidad imperfecta de la autoridad que distribuye el premio en un proceso de cabildeo. También puede ser impuesta por el diseño como en eventos deportivos como el fútbol, el ajedrez, etc, y en concursos promocionales en que la negación de la promoción a todos es un derecho otorgado de un empleador. Mi trabajo se introduce una nueva forma funcional que permite la posibilidad de un empate en un concurso en función de los gastos desembolsados por los concursantes y analiza el juego inducido mediante la asignación de un precio no negativo para el resultado de la eliminatoria. También construyo un conjunto de datos a partir de los cuatro principales ligas de fútbol de Europa, incluyendo información sobre los valores de mercado de los equipos, y el resultado de cada partido durante diez temporadas. Yo uso estos datos como una primera evaluación de los resultados empíricos de las funciones de éxito del concurso de empates en la literatura. De acuerdo a mi forma funcional, la probabilidad de un empate alcanza un máximo cuando concursantes ejercen cantidades iguales de esfuerzo

independientemente de la magnitud de estos esfuerzos; aumenta cuando un jugador con menos esfuerzo aumenta su esfuerzo, y disminuye de otro modo. En el equilibrio único, los jugadores gastan más por el concurso con empates en comparación con la cantidad que gastan para el concurso sin empates, incluso si el premio obtenido en caso de empate es equivalente a perder el concurso. Este resultado implica que los jugadores compiten más incluso si su premio esperado es menor que el uno cuando no hay posibilidad de un empate. El equilibrio también indica que los esfuerzos de equilibrio no dependen en el premio asignado en caso de empate. Por lo tanto, si un diseñador de concurso desea obtener el mayor esfuerzo de la contienda, debe admitir la posibilidad de un empate y asignar premio cero para él. Por otra parte, si hay un jugador limitado en de recursos, un aumento en el premio de empate disminuye los gastos totales gastados por la reducción de los incentivos para el esfuerzo del jugador sin restricciones. La aplicación empírica muestra que mi función de éxito de concurso tiene prometedores resultados en la determinación de la probabilidad de varios resultados posibles de la contienda.

El tercer capítulo es también una aplicación de la economía política de un concurso y considera un conflicto dinámico de dos jugadores en el que, después de una etapa inicial de las decisiones de armado, los jugadores deciden si se debe detener o continuar luchando en cada etapa del conflicto. La guerra no se detiene hasta que ambas partes deciden parar. Las armas son destructivas para las fuerzas rivales y cantidad relativa de fuerzas determina el poder de destrucción y poder de negociación sobre el recurso en cada etapa. En el subjuego que se inicia después de las primeras decisiones de las armas, la guerra no se inicia si los jugadores descartan futuro mucho o uno de los lados tiene una muy grande o una muy pequeña ventaja en fuerzas. Dado que comienza la guerra, la más pequeña es la ventaja relativa de las fuerzas, más largo el conflicto dura. En el equilibrio perfecto en subjuegos de este juego, el único equilibrio es aquel en el que prevalece la paz armada.

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To the unnerving indifference of the Universe...

Chapter 1

Give peace a chance: the effect of ownership and asymmetric information on peace

1.1 Introduction

Rationalist theories explain war as the rational choice of countries (see [Hirshleifer \(1991\)](#), [Skaperdas \(2002\)](#) and surveys by [Garfinkel and Skaperdas \(2007\)](#) and [Jackson and Morelli \(2011\)](#)). This approach shows how factors such as trade, long-term relationships, political bias and the distribution of resources amplify or efface the incentives for war (see [Skaperdas and Syropoulos \(1996\)](#), [Skaperdas and Syropoulos \(2001\)](#), [Garfinkel and Skaperdas \(2000\)](#), [Jackson and Morelli \(2007\)](#) and [Beviá and Corchón \(2010\)](#)).¹

In this paper, we consider a conflict arising between two countries for the control of a resource. Our emphasis will be on the effects of asymmetric information and the distribution of the resource prior to the conflict. To address the second issue, we consider two setups: the Undistributed Resource Game (UR), where countries have no prior ownership of the resource, and the Fully Distributed Resource Game (FDR), where there is a pre-existing distribution of the resource. Examples of the first situation are the Scramble for Africa between all major European powers in 1881-1914 and the Great Game played by British and Russian empires in 1813-1907 for the control of Afghanistan. With respect to the second setup, the distribution of the resource may be achieved by an agreement (such as the treaty of Tordesillas, 1494, in which Spain and Portugal divided South America according to a suggestion made by the Pope), for cultural

¹A forerunner of this approach is [Clausewitz \(1832\)](#), who noted that “War is akin to a card game”.

reasons (language, history), geographical features (a river, a strait, a mountain chain) or by a previous conflict as in the case of Cyprus.²

Other than the initial position, UR and FDR are identical two-stage games. In the first stage, countries decide if they declare war or not. If one of the countries declares war, war occurs in the second stage. If both countries decide not to fight, there is peace and they get zero payoff in UR, and their prior distribution of the resource in FDR.

We first study complete information which serves us as a benchmark case. For UR, war is the only equilibrium outcome. The explanation is that since peace yields the status quo, i.e., zero payoff outcome, a rational country always prefers conflict. For FDR, we show that there is a set of divisions of the resource such that, in equilibrium, both countries will choose peace. The reason is that the status quo for each country is her share of the resource. The possibility of losing this share makes countries reluctant to go to war.

Next, we consider asymmetric information. We assume that country one has private information on how valuable the resource is for her and may have a high or a low valuation (type) of the resource at stake. Country two has only a prior probability that country one is of the high or low type. As an illustration, think that country one has done research on the existence of a valuable resource in a territory under dispute and the results of this research are in the hands of this country only. Also, a country might be uncertain about the willingness of the other country to fight. However, by observing the declaration of her rival, country two has the possibility of inferring the type she faces at war.

We prove for UR that war is the unique Perfect Bayesian equilibrium outcome of the asymmetric information game. For FDR, we find that there are two classes of equilibria. The first class, Peaceful Equilibria, contains equilibria that assigns at least a positive probability to peace. In the second class, all equilibria assign probability one to war.

In the Peaceful Equilibria, there are two kinds of equilibria. In the first, the high type declares war and the low type is peaceful while country two is also peaceful. Consequently, when country one declares war, country two infers she is fighting a high type and subsequently countries play under complete information. There are distributions of the resource for which this equilibrium exists except when the valuations of country two and the low type are low. The reason for the lack of existence is due to the ability of the low type to fake a high type when country two is very weak, because this country will be very insufficiently armed in a conflict. If the low type's valuation is high enough, one can find a distribution of the resource to sustain peace as country two would demand a very low share of the resource.

In the second class of equilibrium, every type and country choose peace. This equilibrium does not exist when all the following possibilities occur jointly:

² For a recent application of mediation to war, see [Horner, Morelli, and Squintani \(2011\)](#).

1. There is a low probability that country one has a high valuation.
2. There is a large dispersion in the possible valuations of country one.
3. The strength and/or valuation of country two is high.

Point 1 is counterintuitive. It says peace can not be achieved when we are close to complete information! The interpretation is that the share of country one is dictated by its high valuation, but when there is a high probability that country one is weak, war looks like a good prospect for country two, especially when the likely low type has a low valuation of the resource (point 2) and country two is powerful or values the resource a lot (point 3). Note that, despite the fact that the high type is unlikely, war occurs with probability one.

We end this section by reviewing the literature. Schelling (1980) and Fearon (1995b) suggested that asymmetric information is a possible cause of war. An early model of war including asymmetric information is by Brito and Intriligator (1985). A thorough discussion of the effects of incomplete information on war is in Jackson and Morelli (2011). They conclude that “If the cost of war is low enough, then country B is better off simply going to war and taking its chances rather than reaching such an unfavorable bargain.” Our findings complete this intuition by showing a list of the causes of war and by highlighting the role of the initial share in the resource. In particular, our results show that *relative* magnitudes matter, namely the dispersion of valuations in country one and the relative strength of country two, and that a low probability that country one has a high valuation is also bad for peace.

The rest of the paper goes as follows. Section 2 spells out the model. Section 3 studies the full information case. Section 4 considers asymmetric information. Finally, Section 5 presents our final comments.

1.2 Model

Two countries dispute a divisible resource which they value in V_1 and V_2 , respectively. In case of war, they incur sunk expenses of g_1 and g_2 . Let p_i be the probability that i obtains the resource after the war.³ p_i is determined by an asymmetric contest success function of the following form:

$$p_i = \begin{cases} \frac{\beta_i g_i}{\sum_{j=1}^2 \beta_j g_j} & \text{if } g_1 + g_2 > 0 \\ \frac{\beta_i}{\beta_i + \beta_j} & \text{if } g_1 + g_2 = 0 \end{cases} \quad (1.1)$$

³ p_i may also be equivalently interpreted as the share of the resource obtained, but for concreteness, we will follow the probabilistic interpretation throughout the paper.

where $\beta_i \in (0, \infty)$ is the productivity of country i in war efforts.⁴ Defining $\theta \equiv \beta_2/\beta_1$ as the relative productivity of country two in war, the contest success function when $g_1 + g_2 > 0$ can be rewritten as:

$$p_1 = \frac{g_1}{g_1 + \theta g_2} \quad \text{and} \quad p_2 = \frac{\theta g_2}{g_1 + \theta g_2} \quad (1.2)$$

Both countries are risk-neutral. In case of war, their expected payoffs are

$$u_1 = V_1 \frac{g_1}{g_1 + \theta g_2} - g_1 \quad \text{and} \quad u_2 = V_2 \frac{\theta g_2}{g_1 + \theta g_2} - g_2, \quad (1.3)$$

The game is played as follows: In the first stage, countries decide to declare war or not. If a country declares war, we will say that this country makes the necessary preparations for war, even though a formal declaration might not be issued. The decisions at this stage are perfectly observed when the next stage begins. In the second stage, if one of the countries declared war, the conflict is waged and payoffs are delivered. Otherwise, we have peace.

Our two-stage model of war avoids the problem arising in one-stage models where starting war is a dominant strategy. The shortcomings of our approach are that we do not allow for surprise attacks and that, once war is prepared, there is no way of achieving peace. We discuss both issues in turn.

The most famous historical surprise attacks -the attacks of Nazi Germany on Russia and Japan on the United States, both in December 1941- ended with the attackers ultimately being defeated. The surprise attack of Japan on Russia in February 1905 was more successful but produced only minimal casualties. In these examples, the fate of war was determined by battles fought later on.⁵ Therefore, it seems that, at least in these historical examples, surprise attacks do not decisively influence the outcome of the conflict. This observation agrees with our model, where the outcome of war relies entirely on the contest success function (1.2) and the expenses made later on, but surprise attacks may play a role in our model in terms of starting a war (see below).

In many actual wars, there was a decision that led inevitably to a conflict: raising an army in 16th-17th century Europe, with no means of supporting itself except by plunder, or accumulating a large number of troops on the border or even a surprise attack which -as we argued before- has no consequence on the outcome of war. It is true that the final spark in some actual wars was somehow random, like in the Spanish-American war of 1898 or the First World War in 1914. However, it may be argued that the decisions made by these countries in previous years made war inevitable sooner or later. Thus our assumption that once war is declared there is no turning back can be regarded as a simplification of a more complex situation involving random elements

⁴In Appendix A, we deal with a more general contest success function and show that our results under complete information do not change qualitatively.

⁵The decisive battles in these wars were Kursk (July-August 1943), Midway (June 1942) and Tsushima (May 1905). See Beevor (2012).

but captures that certain actions, that are irrelevant to the outcome of the conflict, make peace impossible.

We consider two setups. They differ in the status quo prior to the game. In the *Undistributed Resource Game* (henceforth *UR*), countries own nothing out of the resource in dispute. The early race to the conquest of South America between Spain and Portugal at the end of 15th century and so-called *Scramble for Africa* would be examples of this kind of game.⁶ In the *Fully Distributed Resource Game* (henceforth *FDR*), there is a pre-existent full distribution of the resource which defines particular shares for each contestant. In this case, countries engage in war for the full resource, but the payoffs of peace are the shares of the resource given by the initial distribution. The Napoleonic Wars and World Wars I and II are examples of this kind of game.

1.3 Full information case

Assume that all the parameters defining the game are common knowledge between the two countries. We solve the game beginning with the second stage. Assuming that there is war, first order conditions (henceforth FOC) of expected payoff maximization for each country are:

$$\frac{\partial u_1}{\partial g_1} = V_1 \frac{\theta g_2}{(g_1 + \theta g_2)^2} - 1 = 0 = V_2 \frac{\theta g_1}{(g_1 + \theta g_2)^2} - 1 = \frac{\partial u_2}{\partial g_2} \quad (1.4)$$

We verify in Appendix A that these conditions are sufficient. Thus, war efforts are given by (2.7), which implies that

$$\frac{V_2}{V_1} = \frac{g_2}{g_1}. \quad (1.5)$$

Substituting (1.5) into (2.7) and defining $\gamma \equiv \frac{V_2}{V_1}$, we obtain the full information war effort, g_i^F , for $i = 1, 2$.

$$g_1^F = \frac{V_1 \theta \gamma}{(1 + \theta \gamma)^2} \quad (1.6)$$

$$g_2^F = \frac{V_2 \theta \gamma}{(1 + \theta \gamma)^2} \quad (1.7)$$

Substituting expressions (1.6) and (1.7) into (1.3), we obtain the equilibrium payoff for each country:

$$u_1^F = \frac{V_1}{(1 + \theta \gamma)^2} \quad (1.8)$$

$$u_2^F = \frac{V_2 \theta^2 \gamma^2}{(1 + \theta \gamma)^2} \quad (1.9)$$

The payoffs above are strictly positive. Thus, we have the following result:

⁶It can be argued that both historical examples did not end up in war, but by reinterpreting war efforts as the cost of colonization and the resulting p_i 's as the shares in South America/Africa, our analysis is still applicable.

Proposition 1.1. *Under complete information, war is the unique equilibrium outcome of UR.*

Let us now study FDR. Denote the pre-established distribution of the resource in the hands of country one as $\varepsilon \in (0, 1)$. As the resource is fully distributed, the share of country two is $1 - \varepsilon$. For peace to hold in equilibrium, the following condition should hold for the first country.

$$u_1^F = V_1 \frac{1}{(1 + \theta\gamma)^2} \leq \varepsilon V_1. \quad (1.10)$$

which amounts to

$$\frac{1}{(1 + \theta\gamma)^2} \leq \varepsilon \quad (1.11)$$

Performing a similar calculation for the second country, we obtain

$$\varepsilon \leq \frac{1 + 2\theta\gamma}{(1 + \theta\gamma)^2}. \quad (1.12)$$

Then, if peace is achieved, conditions (1.11) and (1.12) imply

$$\frac{1}{(1 + \theta\gamma)^2} \leq \varepsilon \leq \frac{1 + 2\theta\gamma}{(1 + \theta\gamma)^2} \quad (1.13)$$

Note that the right-hand side (henceforth, RHS) of identity (13) is always greater than the left-hand side (henceforth, LHS) of it. Moreover, both sides are positive and less than 1. Thus, we have proved the following:

Proposition 1.2. *Under complete information, there is a set of divisions of the resource given by (1.13), where peace is the unique equilibrium outcome of FDR.*

Equation (1.13) defines the set of shares yielding peace as an equilibrium. Let $x = \theta\gamma$. x measures the magnitude of asymmetries in parties rooted in the fighting power and/or valuation of the resource. As player 2 gets stronger in valuation/fighting power, then $x \rightarrow \infty$. And if player 1 gets stronger in valuation/ fighting power, then $x \rightarrow 0$.

The length of the set for which peace holds as an equilibrium is given by

$$\Phi(x) = \frac{2x}{(1 + x)^2}.$$

$\Phi(x) \rightarrow 0$ as $x \rightarrow 0$, or $x \rightarrow \infty$ and it has a maximum at $x = 1$, i.e., when players are identical. Thus asymmetries between contestants make it harder to find a division that achieves peace as an equilibrium outcome. This is because a stronger contestant would demand a larger share when she has a large possibility of a victory in a decisive battle.

1.4 Asymmetric information

Under asymmetric information, UR and FDR are two-stage games with observed actions and incomplete information where players (countries) simultaneously choose an action at each stage. The types correspond to different possible valuations of the resource. Let \mathbb{V}_i be the type set for country i , $i = 1, 2$. We assume that country one has two possible types denoted by V_H and V_L with $V_H > V_L$. Country two has only one type denoted by V_2 . Now let us introduce the following pieces of notation

$$\gamma = \frac{V_2}{V_H}, \gamma \in (0, 1); \quad \rho = \frac{V_L}{V_H}, \rho \in (0, 1)$$

Note that V_2/V_L equals γ/ρ . Let us assume that $V_H > V_2 > V_L$ which implies:

$$1 > \gamma > \rho. \quad (1.14)$$

Condition (1.14) excludes cases in which country two is “very weak” or “very strong” in terms of valuations of the resource and is introduced mainly, for analytic convenience. Next, we introduce an assumption which guarantees that war expenses are non-negative:

$$\frac{\sqrt{\rho}}{1 - \sqrt{\rho}} \geq \theta\gamma = x \quad (1.15)$$

The LHS of (1.15) is increasing in ρ . The RHS of (1.15) measures the magnitude of asymmetries between countries rooted in the fighting power and/or valuation of the resource between country two and the high type country one (recall that $\theta\gamma$ was denoted by x in the previous section). Thus (1.15) says that the difference between the possible valuations of country one is large compared to the asymmetries between players.

Country two has prior beliefs that country one is of the high type with probability $\pi \in [0, 1]$, and that she is of low type with probability $1 - \pi$.

In the first stage, countries choose an action $a_i \in A = \{P, D\}$ where P is to stay peaceful and D is to declare war. These actions are revealed at the end of this stage. When player one is of type V_H (resp. V_L), her war effort is denoted by g_H (resp. g_L).

At the end of the first stage, player two updates her beliefs about the type of player one, based on the history of actions available in the first stage, denoted by h . We denote \mathcal{H} as the set of all possible histories, i.e., $\mathcal{H} = \{DD, DP, PD, PP\}$.

The posterior belief of player two about the type of player one is $\mu : \mathcal{H} \rightarrow \Delta(\mathbb{V}_1)$, where $\Delta(\mathbb{V}_1)$ is the set of all probability distributions on \mathbb{V}_1 . With a slight abuse of notation, we denote the posterior beliefs as $(\mu(V_H|h), \mu(V_L|h)) = (\mu, 1 - \mu)$. Therefore, country two ex-post believes

that country one is of the high type with probability $\mu \in [0, 1]$, and that she is of the low type with probability $1 - \mu$. As $\mu \leq 1$, from (1.15) it follows that

$$\sqrt{\rho} \geq \mu x [1 - \sqrt{\rho}] \quad (1.16)$$

Let $s = (a, g)$ be a strategy profile consisting of the choice of an action in the first stage and the choice of effort in the second stage. For FDR, we define the following payoff functions for players one and two.

$$u_j(s|h, \mu, V_j) = \begin{cases} V_j \frac{g_j}{g_j + \theta g_2} - g_j & \text{if } h \neq (P, P) \\ \varepsilon V_j & \text{if } h = (P, P) \end{cases} \quad j = H, L \quad (1.17)$$

$$u_2(s|h, \mu, V_2) = \begin{cases} V_2 \left(\mu \frac{\theta g_2}{g_H + \theta g_2} + (1 - \mu) \frac{\theta g_2}{g_L + \theta g_2} \right) - g_2 & \text{if } h \neq (P, P) \\ (1 - \varepsilon) V_2 & \text{if } h = (P, P) \end{cases} \quad (1.18)$$

For UR, the payoffs concerning $h = (P, P)$ are replaced by zero. Otherwise, payoffs are identical to those in (1.17) and (1.18).

We define a Perfect Bayesian Equilibrium (henceforth, PBE) for UR and FDR following Fudenberg and Tirole (1991). For the sake of simplicity, we only consider pure strategies throughout our analysis.

Definition 1.3. For UR and FDR, a PBE in pure strategies is a strategy profile $s^* = (a^*, g^*)$ and posterior beliefs μ such that:

(H) $\forall i = H, L, 2, \forall h \in \mathcal{H}$, and given $\mu \in [0, 1]$;

$$u_i(a^*, g^*|h, \mu, V_i) \geq u_i(a^*, g_i, g_{-i}^*|h, \mu, V_i)$$

(P) $\forall i = H, L, 2$ and given $\pi \in [0, 1]$,

$$u_i(a^*, g^*|\pi, V_i) \geq u_i(a_i, a_{-i}^*, g^*|\pi, V_i)$$

(B1) Denote $\sigma^*(a_1|h, V_1)$ as a degenerate probability distribution over A .

$$\mu' = \frac{\pi \sigma^*(a_1|h, V_H)}{\pi \sigma^*(a_1|h, V_H) + (1 - \pi) \sigma^*(a_1|h, V_L)} \text{ and}$$

$$\pi \sigma^*(a_1|h, V_H) + (1 - \pi) \sigma^*(a_1|h, V_L) > 0$$

(B2) $\forall h$ and $\forall a_2, \hat{a}_2 \in A$, $\mu = \mu(V_H|h = (a_1, a_2)) = \mu(V_H|h = (a_1, \hat{a}_2))$.

(H) and (P) impose each continuation strategy to be a Bayesian equilibrium of the game starting with history h and beliefs corresponding to that history. (H) is the requirement concerning second-stage histories and beliefs, i.e., h , and μ , whereas (P) is the requirement concerning war declaration. (B1) states that the posterior beliefs, given the history in stage two, ought to obey the Bayes' rule whenever possible. We impose that σ^* must be a degenerate probability distribution as we only consider pure strategies. Lastly, (B2) says that, given the signal sent by player one, the posterior beliefs of player two are independent of her action in the first stage.

Throughout this section, we consider separating and pooling equilibria. A separating (resp. pooling) equilibrium is a strategy profile in which different types of player one choose different (resp. the same) actions in the first stage.

We start by analyzing UR. Let us assume for a moment that we have war in the second stage. Then, the expected payoffs of each player given by (1.17) and (1.18), for $h \neq (P, P)$, are strictly concave. Thus FOCs guarantee a unique maximum. For the time being, we disregard non-negativity constraints. Using the reaction functions derived in Appendix A, we get the equilibrium war effort of each country and each player as:

$$g_H^* = \sqrt{\rho} V_H \frac{\left[\frac{\rho}{x} + (1 - \mu)(1 - \sqrt{\rho}) \right] [1 - \mu(1 - \sqrt{\rho})]}{\left[\frac{\rho}{x} + 1 - \mu(1 - \rho) \right]^2} \quad (1.19)$$

$$g_L^* = \rho V_H \frac{\left[\frac{\rho}{x} - \mu\sqrt{\rho}(1 - \sqrt{\rho}) \right] [1 - \mu(1 - \sqrt{\rho})]}{\left[\frac{\rho}{x} + 1 - \mu(1 - \rho) \right]^2} \quad (1.20)$$

$$g_2^* = \frac{\rho V_H}{\theta} \left[\frac{1 - \mu(1 - \sqrt{\rho})}{\frac{\rho}{x} + 1 - \mu(1 - \rho)} \right]^2 \quad (1.21)$$

Note that g_H^* and g_2^* are both strictly positive and by (1.16), g_L^* also is strictly positive. Then, substituting (1.19)-(1.21) into (1.17) and (1.18), we get the following equilibrium payoffs for each player and type.

$$u_H^* = V_H \left[\frac{\frac{\rho}{x} + (1 - \mu)(1 - \sqrt{\rho})}{\frac{\rho}{x} + 1 - \mu(1 - \rho)} \right]^2 \quad (1.22)$$

$$u_L^* = \rho V_H \left[\frac{\frac{\rho}{x} - \mu\sqrt{\rho}(1 - \sqrt{\rho})}{\frac{\rho}{x} + 1 - \mu(1 - \rho)} \right]^2 \quad (1.23)$$

$$u_2^* = V_2 \frac{[1 - \mu(1 - \rho)] [1 - \mu(1 - \sqrt{\rho})]^2}{\left[\frac{\rho}{x} + 1 - \mu(1 - \rho) \right]^2} \quad (1.24)$$

Note that all payoffs above are strictly positive.

Lemma 1.4. *There is no PBE for UR in which player two is peaceful and at least one type of player one is peaceful.*

Proof. Assume first $a = (P, P, P)$. By requirement (H), $g = (g_H^*, g_L^*, g_2^*)$ are given by (1.19)-(1.21) for any system of beliefs, which in turn implies $u_i^* > 0, \forall i$. Denote \hat{h} as the alternative history in which player two chooses D . Then, using (B2), if player two deviates to D , she obtains $u_2^* > 0$.

Now assume that $a = (D, P, P)$. In this profile, $u_2(a^*, g^*|h, v_2, \mu) = \pi u_2^*$. Using (B2), player two gets u_2^* if she deviates, which is larger. The proof concerning the profile $a^* = (P, D, P)$ is identical to the one above. \square

Lemma 1.5. *There is no separating equilibrium in which player two declares war.*

Proof. Assume $a = (D, P, D)$ or $a = (P, D, D)$. Take the former profile, and assume it is part of the PBE. The consistent beliefs are $\mu = 1$, and there is war with probability one. The payoff of the high type is given by substituting $\mu = 1$ into (1.22), whereas the deviation payoff is given by substituting $\mu = 0$ into (1.22). Thus, for a high type player not to deviate:

$$\left[\frac{\frac{\rho}{x}}{\frac{\rho}{x} + \rho} \right]^2 > \left[\frac{\frac{\rho}{x} + 1 - \sqrt{\rho}}{\frac{\rho}{x} + 1} \right]^2 \iff \frac{\frac{\rho}{x} + \rho}{\frac{\rho}{x} + 1} > \sqrt{\rho} \quad (1.25)$$

The payoff of the low type implied by the profile is given by substituting $\mu = 0$ into (1.23), whereas the deviation payoff is given by substituting $\mu = 1$ into (1.23). Therefore, for a low type player not to deviate:

$$\left[\frac{\frac{\rho}{x}}{\frac{\rho}{x} + 1} \right]^2 \geq \left[\frac{\frac{\rho}{x} + \rho - \sqrt{\rho}}{\frac{\rho}{x} + \rho} \right]^2 \iff \frac{\frac{\rho}{x} + \rho}{\frac{\rho}{x} + 1} \leq \sqrt{\rho} \quad (1.26)$$

which contradicts (1.25). The proof concerning $a = (P, D, D)$ is identical. \square

Proposition 1.6. *War is the unique PBE outcome of UR.*

Proof. By lemmata 4 and 5, there is not any peaceful PBE. Thus, if we conclude that at least one of the two remaining profiles, (D, D, P) or (D, D, D) , is an equilibrium, we prove the proposition.

Now assume that $a = (D, D, P)$. War occurs. By (B2), player two is indifferent between the two actions, whereas if any type deviates, players achieve the zero payoff outcome. As war always pays off a positive expected amount, no type will have any incentive to deviate. \square

Proposition 6 says that the results obtained in UR under asymmetric information parallel the ones from UR under complete information. Specifically, peace is impossible in this setting under complete or asymmetric information.

We now study FDR under asymmetric information. First, we show that there is a unique separating equilibrium. In particular, by lemmata 7 and 8 below, we first eliminate some of

the candidate equilibrium profiles. Then, in proposition 9, we confirm that the only remaining strategy profile is a PBE.

Lemma 1.7. *There is no separating equilibrium for FDR in which player two declares war.*

Proof. Assume $a = (D, P, D)$ or $a = (P, D, D)$. War occurs. Thus, the conditions are the same as (1.25) and (1.26) and the proof concerning FDR is the same as the proof concerning UR. \square

Lemma 1.8. *There is no separating equilibrium for FDR under asymmetric information in which the high type is peaceful, and the low type declares war when player two is peaceful.*

Proof. $a = (P, D, P)$. The consistent beliefs imply $\mu = 0$. Substituting $\mu = 0$ into equations (1.22) and (1.23), we find the equilibrium payoffs of war for the high type and the low type, respectively, as follows:

$$V_H \left[\frac{\frac{\rho}{x} + 1 - \sqrt{\rho}}{\frac{\rho}{x} + 1} \right] \quad \text{and} \quad V_L \left[\frac{\frac{\rho}{x}}{\frac{\rho}{x} + 1} \right]$$

For this profile to be an equilibrium, the high type should be better off under peace:

$$\varepsilon \geq \left[\frac{\frac{\rho}{x} + 1 - \sqrt{\rho}}{\frac{\rho}{x} + 1} \right]$$

Moreover, the low type has to be better off declaring war

$$\varepsilon \leq \left[\frac{\frac{\rho}{x}}{\frac{\rho}{x} + 1} \right]$$

which constitutes a clear contradiction. \square

Proposition 1.9. *There is a unique separating equilibrium of FDR in which country 2 and the low type are peaceful as the high type declares war if and only if:*

$$\left[1 - \frac{x}{\sqrt{\rho}(1+x)} \right]^2 \leq \varepsilon \leq \frac{\rho(\rho+2x)}{(\rho+x)^2} \quad (1.27)$$

Proof. Assume $a = (D, P, P)$ and condition (1.15) is satisfied. The consistent beliefs imply $\mu = 1$. Then, condition (H) implies the following ex-post payoffs for the high type and player two, respectively:

$$u_H = \frac{V_H}{(1+x)^2} \quad \text{and} \quad u_2 = \frac{V_2 x^2}{(1+x)^2}$$

If player one is a low type, peace is obtained with the following ex-post payoffs.

$$u_L = \varepsilon \rho V_H, \quad u_2 = (1 - \varepsilon) V_2$$

For a to be a part of the PBE, a high type ought to choose peace:

$$u_H \geq u_H^D = \varepsilon V_H \iff \frac{1}{(1+x)^2} \geq \varepsilon \quad (1.28)$$

Also, a low type player ought not to deviate to declaring war. Given a , player two will act as if she were facing a high type. Substituting $\mu = 1$ into (1.23), we get:

$$u_L^D = \rho V_H \left[\frac{\frac{\rho}{x} - \sqrt{\rho}(1 - \sqrt{\rho})}{\frac{\rho}{x} + \rho} \right]^2$$

The condition for the low type player is $u_L = \varepsilon \rho V_H \geq u_L^D$, which reduces to

$$\left[\frac{\frac{\rho}{x} - \sqrt{\rho}(1 - \sqrt{\rho})}{\frac{\rho}{x} + \rho} \right]^2 = \left[1 - \frac{x}{\sqrt{\rho}(1+x)} \right]^2 \leq \varepsilon \quad (1.29)$$

Finally, player two should not deviate. The expected payoff of player two is:

$$u_2 = \pi \frac{V_2 x^2}{(1+x)^2} + (1-\pi) V_2 (1-\varepsilon)$$

whereas the deviation payoff given the belief $\mu = 1$ is:

$$u_2^D = \pi \frac{V_2 x^2}{(1+x)^2} + (1-\pi) \frac{V_2 (x/\rho)^2}{(1+(x/\rho))^2}$$

Hence, the condition for player two is $u_2 \geq u_2^D$, implying:

$$\frac{1+2(x/\rho)}{(1+(x/\rho))^2} = \frac{\rho(\rho+2x)}{(\rho+x)^2} \geq \varepsilon \quad (1.30)$$

Combining (1.28)-(1.30), the condition for a being a part of the PBE is:

$$\left[1 - \frac{x}{\sqrt{\rho}(1+x)} \right]^2 \leq \varepsilon \leq \min \left\{ \frac{\rho(\rho+2x)}{(\rho+x)^2}, \frac{1}{(1+x)^2} \right\}$$

Note that the LHS of the condition above is less than the second term in the RHS of it. Hence, we drop the condition concerning the high type, implying that the sufficient condition for the existence of the equilibrium is given by (1.27). By lemmata 7 and 8, there is no other separating equilibrium. Combining conditions (1.28)-(1.30), we get the result. \square

Now we look for pooling equilibria. As mentioned before, in a pooling equilibrium, both types of player one either declare war or stay peaceful.

Proposition 1.10. *For FDR, given beliefs $\mu(V_H|h) = \pi$ where $h \in \{(D, \cdot), (P, \cdot)\}$ and g satisfies (H);*

(i) $a = (D, D, D)$ is a part of the pooling equilibrium if off-the-path beliefs $\mu(V_H|(P, \cdot)) = \mu^o$ satisfy:

$$\pi \leq \mu^o \text{ if } x \geq \sqrt{\rho}, \text{ and } \pi > \mu^o \text{ if } x < \sqrt{\rho} \quad (1.31)$$

(ii) $a = (P, P, D)$ is a part of the pooling equilibrium if off-the-path beliefs $\mu(V_H|(D, \cdot)) = \mu^o$ satisfy:

$$\begin{aligned} \pi \leq \mu^o \text{ if } x \geq \sqrt{\rho}, \quad \pi > \mu^o \text{ if } x < \sqrt{\rho} \\ \text{and} \\ 1 - \varepsilon \leq \frac{[1 - \pi(1 - \rho)][1 - \pi(1 - \sqrt{\rho})]^2}{\left[\frac{\rho}{x} + 1 - \pi(1 - \rho)\right]^2} \end{aligned} \quad (1.32)$$

(iii) $a = (D, D, P)$ is a part of the pooling equilibrium if

$$\varepsilon \leq \left[\frac{\frac{\rho}{x} - \pi\sqrt{\rho}(1 - \sqrt{\rho})}{\frac{\rho}{x} + 1 - \pi(1 - \rho)} \right]^2 \quad (1.33)$$

(iv) $a = (P, P, P)$ is a part of the pooling equilibrium if off-the-path beliefs $\mu(V_H|(D, \cdot)) = \mu^o$ satisfy:

$$\begin{aligned} \varepsilon \geq \left[\frac{\frac{\rho}{x} + (1 - \mu^o)(1 - \sqrt{\rho})}{\frac{\rho}{x} + 1 - \mu^o(1 - \rho)} \right]^2 \\ \text{and} \\ 1 - \varepsilon \geq \frac{[1 - \pi(1 - \rho)][1 - \pi(1 - \sqrt{\rho})]^2}{\left[\frac{\rho}{x} + 1 - \pi(1 - \rho)\right]^2} \end{aligned} \quad (1.34)$$

Proof. (i) For (D, D, D) , deviation payoffs for each type are given by (1.22) and (1.23) for $\mu = \mu^o$. Then, condition (P) implies that off-the-equilibrium beliefs have to obey:

$$\begin{aligned} \frac{\frac{\rho}{x} + (1 - \mu^o)(1 - \sqrt{\rho})}{\frac{\rho}{x} + 1 - \mu^o(1 - \rho)} &\leq \frac{\frac{\rho}{x} + (1 - \pi)(1 - \sqrt{\rho})}{\frac{\rho}{x} + 1 - \pi(1 - \rho)} \\ \frac{\frac{\rho}{x} - \mu^o\sqrt{\rho}(1 - \sqrt{\rho})}{\frac{\rho}{x} + 1 - \mu^o(1 - \rho)} &\leq \frac{\frac{\rho}{x} - \pi\sqrt{\rho}(1 - \sqrt{\rho})}{\frac{\rho}{x} + 1 - \pi(1 - \rho)}, \end{aligned}$$

for the high and the low type, respectively. Simplifying the inequalities above, we see that they hold iff $(\pi - \mu^o)(\sqrt{\rho} - x) \geq 0$, which is condition (1.31).

(ii) For (P, P, D) , the first two conditions are equivalent to the conditions for part (i). Given $\mu^o = \pi$ by (B2), for player two not to deviate to peace, $u_2^* \geq (1 - \varepsilon)V_2$, implying the condition.

(iii) If (D, D, P) is a part of the PBE, $u_H^* \geq \varepsilon V_H$, and $u_L^* \geq \varepsilon \rho V_H$. By (1.22) and (1.23), $u_L^* \geq \varepsilon \rho V_H \Rightarrow u_H^* \geq \varepsilon V_H$ because:

$$\frac{\frac{\rho}{x} - \mu\sqrt{\rho}(1 - \sqrt{\rho})}{\frac{\rho}{x} + 1 - \mu(1 - \rho)} \leq \frac{\frac{\rho}{x} + (1 - \mu)(1 - \sqrt{\rho})}{\frac{\rho}{x} + 1 - \mu(1 - \rho)}$$

Therefore, the binding condition is for the low type. By (B2), player two is indifferent between her two actions.

(iv) If (P, P, P) is a part of the PBE, $u_H^* \leq \varepsilon V_H$, $u_L^* \leq \varepsilon \rho V_H$, and $u_2^* \leq (1 - \varepsilon) V_2$. As $u_H^* \geq u_L^*$ by (1.22) and (1.23), $u_H^* \leq \varepsilon V_H \Rightarrow u_L^* \leq \varepsilon \rho V_H$ because:

$$\frac{\frac{\rho}{x} + (1 - \mu^o)(1 - \sqrt{\rho})}{\frac{\rho}{x} + 1 - \mu^o(1 - \rho)} \geq \frac{\frac{\rho}{x} - \mu^o \sqrt{\rho}(1 - \sqrt{\rho})}{\frac{\rho}{x} + 1 - \mu^o(1 - \rho)}$$

Thus, the binding condition is for the high type. By (B2), if player two declares war, she has the same beliefs; i.e., $\mu^o = \pi$. Thus,

$$1 - \varepsilon \geq \frac{[1 - \pi(1 - \rho)][1 - \pi(1 - \sqrt{\rho})]^2}{\left[\frac{\rho}{x} + 1 - \pi(1 - \rho)\right]^2}$$

□

We now focus on two equilibria in which peace is the possible outcome: The separating equilibrium in Proposition 9, where only the high type declares war, and the pooling equilibrium in Proposition 10 part (iv), in which all types and all players are peaceful.

First, we consider the pooling equilibrium. Note that the pooling equilibrium implies probability one for peace. Therefore, we choose to name this equilibrium Definitely Peaceful Equilibrium (DPE). This equilibrium holds if (1.34) is satisfied.

We assume that off-the-equilibrium path beliefs are the prior beliefs; i.e., $\mu(V_H|(D, \cdot)) = \mu^o = \pi$. Other assumptions about those beliefs are possible, but we focus on this case for the sake of simplicity. Thus, the two conditions of (1.34) boil down to:

$$1 - \frac{[1 - \pi(1 - \rho)][1 - \pi(1 - \sqrt{\rho})]^2}{\left[\frac{\rho}{x} + 1 - \pi(1 - \rho)\right]^2} \geq \varepsilon \geq \left[\frac{\frac{\rho}{x} + (1 - \pi)(1 - \sqrt{\rho})}{\frac{\rho}{x} + 1 - \pi(1 - \rho)} \right]^2 \quad (1.35)$$

Rearranging (1.35), we reach the following condition for peace to hold in equilibrium:

$$\frac{1}{x} \geq \frac{(1 - \pi)(1 - \sqrt{\rho})^2 \left\{ (1 - \pi)^2 - \pi^2 \rho \right\}}{2\rho\sqrt{\rho} \left[1 - \pi(1 - \sqrt{\rho}) \right]} \quad (1.36)$$

The RHS of (1.36) will be denoted as $F(\pi, \rho)$. We have two cases:

If $(1 - \pi)^2 / \pi^2 \leq \rho$, the RHS of (1.36) is non positive. Therefore, there is a distribution of the resource that achieves peace. This case arises if ρ is very close to one (so both types have similar valuations) or when π is very close to one, so the probability of a high type is overwhelming. In

both cases, the uncertainty is small. Note that this case is compatible with our assumption in (1.15) iff $\sqrt{\rho} > \max\{\frac{1-\pi}{\pi}, \frac{x}{1+x}\}$.

If $(1-\pi)^2/\pi^2 > \rho$, the RHS of (1.36) is positive and things are more involved. Firstly, note that this case is compatible with (1.15) iff $\frac{1-\pi}{\pi} \geq \sqrt{\rho} \geq \frac{x}{1+x}$ which implies that $1 \geq \pi + x + 2\pi x$; i.e., π , and x cannot be very large at the same time. Now, let us discuss the role of each parameter separately.

The role of x is clear. When the strength of country two or her valuation is low, peace holds because this country could only demand a small share and this is always feasible.

To find the role of π , we partially differentiate the right-hand side of (1.36) with respect to π and find the following expression:

$$\text{sign} \frac{\partial F(\pi, \rho)}{\partial \pi} = \text{sign}[-2 + 2\pi(1 + \sqrt{\rho}) - \sqrt{\rho}]. \quad (1.37)$$

It is easy to see that

$$\text{sign} \frac{\partial F(\pi, \rho)}{\partial \pi} < 0 \Leftrightarrow \pi < \frac{2 + \sqrt{\rho}}{2(1 + \sqrt{\rho})}$$

but the last inequality is implied by $(1-\pi)^2/\pi^2 > \rho$. Thus $F(\pi, \rho)$ decreases with π and we conclude that a large value of π gives peace a good chance. This is very intuitive because uncertainty is very small. However, a sufficiently low value of π makes war very likely. Indeed, when $\pi \simeq 0$, the condition for war is

$$2 \frac{\rho\sqrt{\rho}}{(1 - \sqrt{\rho})^2} < x. \quad (1.38)$$

In order to show that (1.38) is compatible with our assumptions, we provide an example. Let $x = 2/5$ (e.g., $\theta = 1$ and $\gamma = 2/5$), and $\rho = 1/9$, so $1 > \gamma > \rho$. By (1.15), the low type puts a positive effort in war. However, we will have war at $\pi \rightarrow 0$, as $2\rho\sqrt{\rho}/(1 - \sqrt{\rho})^2 = 1/6$.

Note that the LHS of (1.38) is increasing in ρ , so in this case, war arises as a combination of a low probability of country one being of the high type, a large valuation and/or strength of country two and a small valuation for the low type country one. In all these cases, the share inducing the high type to be peaceful looks too expensive for country two, which has a good chance of winning a sizable chunk of the prize by going to war.

It is remarkable that despite the fact that when $\pi = 0$, war cannot happen in FDR (see Proposition 2), war is perfectly possible when π is very close to zero. Even more, war is more likely the closer the value of π is to zero. This reveals an interesting discontinuity in the prevention of war.

Finally, it is easily seen that $\frac{\partial F(\pi, \rho)}{\partial \rho} < 0$ for $F(\pi, \rho) \geq 0$. Again, if ρ is close to 1, $F(\pi, \cdot)$ is close to 0 and peace has a fair chance. This is because we are close to complete information. However, if ρ is low, $F(\pi, \cdot)$ could be large and make peace impossible. This is because the low type will make very little effort in war and will lose it with a high probability, but its share in the resource is determined by the high type, so war looks like a good prospect for country two.⁷

Summing up, from the discussion above, we learn that the failure of the existence of DPE for FDR is due to:

1. Large relative strength and/or valuation of country two.
2. Low probability that country one has a high valuation.
3. Large dispersion in the possible valuations of country one.

The mechanism under which war occurs is that country one appears, in expected terms, as a weak opponent in war, but peace can only be avoided when the share of this country is given by the characteristics of the high type.

Now we consider the separating equilibrium in Proposition 9. Realize that in this equilibrium, war takes place with probability π . Hence we call this equilibrium Possibly Peaceful Equilibrium (PPE). The relevant condition for the existence of this equilibrium is given by:

$$\left[1 - \frac{x}{\sqrt{\rho}(1+x)}\right]^2 \leq \frac{\rho(\rho+2x)}{(\rho+x)^2} \quad (1.39)$$

In Appendix A, we show that the inequality (1.39) holds when ρ and x are not very small. For example, for $\rho = 0.01$, $\gamma = 0.02$ and $\theta = 1$, (1.39) fails to hold. The explanation is that when x is very small, the effort of player two is biased to small values when she observes a declaration at the beginning of the second stage. However, ex-ante, because of the probability of a very weak opponent, she still demands some considerable share of the resource. Thus, if the low type deviates and declares war, she can benefit from the beliefs of player two, who believes that she fights a powerful rival.

We also show in Appendix A that if x is large, implying that player two is strong, an increase in ρ makes the possible partitions of the resource that sustain PPE larger. This result partially parallels the results obtained under complete information. Realize that in PPE, the peaceful players are both the low type and player two. Thus if their strength/valuation are similar, the possible partitions that satisfy both parties is a large set, given that the partition does not satisfy the high type. However, when x is small, an increase in ρ makes the possible partitions that allow for PPE to be smaller, as when country two is very weak and there is not much difference between a high and a low type, the corresponding partitions are a small set. Note, however,

⁷Even though condition (1.15) gets harder to be obtained as $\rho \rightarrow 0$, it can be shown that there are values of other parameters that allow condition (1.15) to be fulfilled.

that there are possible partitions supporting PPE, when country two is very weak, because she only demands a small amount of resources, unlike the case where ρ and x are both low.

Note that if $\varepsilon \geq 1/(1+x)^2$, PPE fails to exist, as the high type would be rich to risk a war. Combined with π being sufficiently low, DPE also fails to exist, implying that in the remaining equilibria war occurs.

1.5 Conclusions

In this paper, we studied the role of information and resource ownership in conflicts. In order to conform with our initial motivation we only mentioned warfare throughout the paper. However, we can also apply our model to litigation. An example is divorce proceedings in which a previous partition of the resource, e.g. total wealth of the couple and full/partial custody rights, may or may not exist. This situation would be a straightforward application of FDR and UR.

We now summarize our results. In the benchmark case of complete information, when the resource is not distributed, war always occurs. When the resource is distributed, war is a consequence of the interplay between asymmetries of contestants (embedded in x) and the distribution of ownership (ε). When the latter does not reflect the former adequately, war occurs. The good news is that war can always be stopped by the appropriate distribution of resources. These conclusions agree with some theories about wars in 17th century Europe when the Spanish empire was militarily weak but owned large territories and France was militarily powerful and contended part of the territories owned by Spain (a similar case occurred in the 20th century with the British Empire and Germany as players). Our theory, contrary to some theories of war, stresses the role of relative asymmetries; it is not the size of the prize that triggers war but the relative distribution of it in relationship to the relative strength of the armies.⁸

Under asymmetric information, when the resource is not distributed, the outcome is always war. Thus, asymmetric information does not alter the picture when there is no prior ownership. However, when the resource is distributed, informational asymmetries make a difference regarding war or peace. We distinguish between war in a pooling equilibrium, in which both types declare war, and war in a separating equilibrium, in which only the high type declares war.

In a pooling equilibrium, if the uninformed country assigns a low probability that the informed country has a high type, war always occurs. This is noteworthy because it shows that a little asymmetric information may cause war. Apparently inexplicable facts like the Nazi invasion of the USSR might be explained by this mechanism. A weak USSR owned “too much land” with respect to the military power the Nazis expected from her. The German beliefs of a weak USSR

⁸ “Unsustainable practices led toagriculturally marginal lands having to be abandoned again. Consequences for society included ... wars” Diamond (2005) p. 15.

were justified by the “Great Purge” in which Joseph Stalin disposed of many capable military leaders, resulting in a significant weakening of the Red Army. A similar case might be made with respect to the attack of the Japanese Empire on Pearl Harbor. Japanese leaders believed that the United States had too much influence in the Pacific relative to their expected willingness to fight (which they thought was low). The beliefs of the Japanese elite were justified by the fact that in 1941 the US economy was still recovering from the Great Depression.

In a separating equilibrium only the high type declares war. This is reminiscent of the Russo-Japanese war (1904-05), where Japan declared war on Russia. The cause of the war was, for one, that Russia did not want to recognize the Japanese influence in Korea and, furthermore, the belief of many influential Russians that (despite the complete victory of Japan over China in the 1894-5 war) “Japan is not a country that can issue an ultimatum to Russia” (see Ferguson (2006, Chap. 2)). In any case, deeper research is needed on the applications of the contest model to history; see Hoffman (2012) for a recent entry on this. Our paper just provides a theoretical model and sketches some possible applications.

In order to make the model tractable, we have made a number of assumptions and left aside some questions that we discuss now.

1. We assumed that after war, no compensation is paid by the loser, but there are historical examples in which compensations were paid, i.e., the Franco-Prussian War (1870-1), World War I (1914-1918), etc. Farmer and Pecorino (1999) have shown that when the loser has to pay the expenses of the winner, total expenses might skyrocket because the winner pays nothing. It would be interesting to know how war may arise in this case. Given that payoffs under war are smaller than under no compensation, intuition suggests that peace can be even more likely in this case than under no compensation.
2. Another extension would be to consider a political bias as in Jackson and Morelli (2007). In this paper, the agent running a country might receive high profits from the victory but pay only a fraction of the cost of war. In this case, if the bias is sufficiently large, peace cannot hold.
3. Finally, bargaining for the distribution of the resource usually takes several rounds. It would be interesting to model the distributions of the resource that could be achieved by bargaining as in the model of Rubinstein (1982).

We hope that our paper sheds light on the powers and limitations of achieving peace by means of the distribution of resources and pinpoints the cases in which achieving peace by this means is bound to fail and other measures like direct UN intervention have to be taken.

Chapter 2

Contests with ties and an application to soccer

2.1 Introduction

A contest defines a competition where participants spend costly efforts to obtain a prize. The prize takes the form of territory and natural resources in military campaign ([Hirshleifer \(1991\)](#)); medals and points in sports ([Szymanski \(2003\)](#)); market shares and sales revenues in marketing ([Schmalensee \(1976\)](#)) and costly efforts also differ accordingly with respect to the context in question. Despite the tools and the stakes of the competitions differ, the very nature of the competition is quite similar. As a result, all the papers mentioned above use the models implied by a *Contest Success Function* (CSF, hereinafter). A CSF is a function that assigns probabilities of winning a contest or shares of the prize for each contestant in terms of the individual and total efforts spent in the competition.

CSFs have been exclusively developed in the context of two outcomes, i.e. ‘win’ and ‘lose’, taking either the ratio form due to [Tullock \(1980\)](#), which was later axiomatized by [Skaperdas \(1996\)](#) and [Clark and Riis \(1998\)](#), or the difference form due to [Hirshleifer \(1989\)](#).

However, ties are naturally existent or imposed by design in various contests and their occurrence is generally a better outcome than a loss for every active contestant. In military conflicts a tie is considered as an impasse between belligerents and depending on the nature of the conflict and the destruction inflicted on the conflicted resource it may bind a better outcome for an adversary than a defeat¹. History also exemplifies the inherence of impasses in military conflicts with Battle of Kadesh (1259 BC), the Korean War (1950-1953), the Sand War (1963), and the Iran-Iraq War (1980-1988).

¹[Garfinkel and Skaperdas \(2007\)](#) and [Blavatsky \(2010\)](#) define this outcome as each side having the identical bargaining power in the aftermath of the conflict.

The occurrence of a tie is imposed by design in some contests. Obvious examples are sports competitions such as soccer, chess, and cricket where the outcome of a tie grants each side a prize that is more valuable than losing the contest. More obscure though solid examples include promotional contests and labor-market competitions where the outcome of a tie is simply equivalent to the outcome of losing in terms of the prize allocated. It is not uncommon to see in job openings the statement of an employer reserving his right to deny the position to all candidates. This setting is also in line with the imperfect commitment of the prize granting institution in lobbying as [Kahana and Nitzan \(1999\)](#) point out.

Despite these examples, there is very recent and few focus on contests ties. [Blavatskyy \(2010\)](#) proposes a CSF for ties with a microeconomic underpinning which has later been stochastically derived by [Jia \(2012\)](#). In his specification, the probability of a tie is a decreasing function of the total efforts spent in the contest. Even though [Blavatskyy \(2010\)](#) argues that it is intuitive for a tie to take place rarely when players spend large efforts, there are also cases which may not fit his argument. [Peeters and Szymanski \(2012\)](#) argue the implausibility of this form for sports and introduce a CSF where probability of a tie is determined by the relative difference of efforts. However, they did not analyze this form in depth, but rather used for empirical motives. Finally, [Jia \(2012\)](#) stochastically axiomatizes a functional form, which is first introduced by [Rao and Kupper \(1967\)](#), and that implies that the probability of a tie is a function of the difference between squared effective efforts of the players. [Jia \(2012\)](#) axiomatizes this form by introducing a ‘coarseness parameter’, capturing the lack of clarity of the outcome when the difference in efforts does not exceed a fixed threshold.

In this paper, I introduce a new CSF that generalizes the ratio form CSF of [Tullock \(1980\)](#) by adding a “tie-proneness” parameter that measures the exposure of a contest to result in a tie. This CSF implies that the probability of a tie achieves a maximum whenever the players exert equal efforts in the contest, and the maximized value is the same no matter how large the equal efforts are. Moreover, the probability of a tie is increasing in the effort level of a player exerting a relatively small effort and decreasing otherwise.²

I later analyze the two-player game induced by the CSF. I model the contest as a game including two possibilities in terms of the final outcome: either one player wins or nobody wins (a tie). In case of a win, the winner obtains a positive prize and the other obtains nothing. In case of a tie, both players obtain a ‘tie-prize’ that is strictly less valuable than the ‘win-prize’ but larger than the ‘lose-prize’.³ Under some mild sufficiency conditions, the game possesses a symmetric and unique pure strategy Nash equilibrium and the value of the tie-prize is irrelevant in the magnitude of the equilibrium efforts spent.

²Our CSF shares some properties with the one found in [Rao and Kupper \(1967\)](#) and [Jia \(2012\)](#). However, the functional form we use is completely different. Consequently, the two games implied are strategically inequivalent and results obtained differ remarkably.

³This specification is reminiscent of multiple prize contests in [Clark and Riis \(1998\)](#) and [Sisak \(2009\)](#)

Next, in order to assess the behavioral differences of contestants when tie outcome is existent, I compare two contests: the one with ties and the one without. I first find that in equilibrium players spend more effort in the former if tie-proneness is sufficiently low even when tying is equivalent to losing in terms of the prize obtained. Thus, the competition is more intense when ties are possible. This apparently counter-intuitive result is because of the best response of a player being larger for a sufficiently small rival effort. For the former competition, the marginal value of an additional effort in response to a small rival effort is larger as this additional effort not only marginally decreases the rival possibility of winning but also decreases any possibility of a tie. I also find that the equilibrium effort has a unique maximum in tie-proneness and.

The results above suggest that allowing for ties benefits a contest designer who wishes to maximize the total amount of effort in the contest. By costlessly introducing the outcome of a tie and allocating zero prize for this outcome he may achieve a larger total effort from the contest as opposed to a contest without any tie. Then, theoretically, one may expect to see players being more competitive and spending more in military conflicts and labor-market contests when tying is a possibility.

I actually show that, surprisingly, the tie-prize has an adverse impact on total efforts spent in the contest when one of the contestant is constrained in the resources he may deploy for the contest. In this case, an increase in the value of the tie-prize leads to decrease in the effort spent by the unconstrained player, as the increased value of tying reinforces incentives for tying for which having similar efforts is important. Even though, the same incentives are also reinforced for the constrained player to achieve a similar effort level he has to increase his, which is impossible by definition.

Finally, using a dataset I build from four major European soccer leagues, I assess the empirical performance of four CSFs: the three mentioned before including a benchmark random choice model. Using the market values of teams as a proxy for effort, I estimate the parameters of each model using maximum likelihood, and I use [Vuong \(1989\)](#) test to compare the performance of the four CSFs. To my knowledge, this is the first comparison for the empirical performance of CSFs including tie as an outcome. The comparison yields favoring results for our CSF and the CSF of [Rao and Kupper \(1967\)](#) and [Jia \(2012\)](#). Moreover, the estimated parameters for our model suggest that if all the clubs in the league spent the same amount of effort, out of all matches played in a season, approximately half of the games would be won by the home teams and the rest would be divided almost equally between ties and visiting teams winning. The results also suggest that there are major differences across national soccer leagues in terms of the importance of investment and playing at home.

In Section 2, I define the CSF and the game implied. In Section 3, I analyze the existence of a Nash equilibrium for this game and I show the properties of the equilibrium and compare the

two contests. In Section 4, I describe the estimation method, the data and present the results obtained and I discuss these results and I conclude with Section 5.

2.2 Model

Two risk neutral players in the set $N = \{1, 2\}$ compete for a prize whose value is normalized to one, without loss of generality.⁴ To obtain the prize, players simultaneously spend costly effort $x_i \geq 0$ with a cost function $c(x_i) = x_i$. The contest has two possible outcomes: either one player wins or the contest ends in a tie. If a player wins, she gets the prize and the other obtains nothing. If the contest ends in a tie each player obtains a prize $b \in [0, 1)$.

Given a vector of efforts $x = (x_1, x_2)$, I define the probability of player $i \in N$ winning the contest by the CSF:

$$p_i(x) = \begin{cases} \left(\frac{f(x_i)}{\sum_{j \in N} f(x_j)} \right)^k & \text{if } x_j > 0 \text{ for some } j \in N \\ \frac{1}{2^k} & \text{if } x_j = 0 \forall j \in N \end{cases} \quad (2.1)$$

On the other hand, the probability of a tie is:

$$p_t(x) = \begin{cases} 1 - \frac{\sum_{j \in N} f(x_j)^k}{\left(\sum_{j \in N} f(x_j) \right)^k} & \text{if } x_j > 0 \text{ for some } j \in N \\ 1 - \frac{1}{2^{k-1}} & \text{if } x_j = 0 \forall j \in N \end{cases} \quad (2.2)$$

The function $f : \mathbb{R}_+ \cup \{0\} \rightarrow \mathbb{R}_+ \cup \{0\}$ is strictly increasing, continuous, twice differentiable, concave and satisfies the Inada conditions $f(0) = 0$, $\lim_{x \rightarrow 0} f'(x) = \infty$, $\lim_{x \rightarrow \infty} f'(x) = 0$. I call $f(x_i)$ the effective effort of player $i \in N$.

The novelty of the CSF is the parameter $k \geq 1$, which I call the “tie-proneness” parameter as ties become more prevalent if k increases. Moreover if $k = 1$, $p_t(x) = 0$ and $\sum_{j \in N} p_j(x) = 1$. Thus, one obtains the usual Tullock CSF. In the following lemma I show that probabilities assigned by our CSF are well-defined.

Lemma 2.1. $k \geq 1$ implies $1 \geq p_i \geq 0$, $\forall i \in N$ and $\sum_{l \in M} p_l = 1$, where $M = N \cup \{t\}$.

⁴Even though the CSF generalizes to more than two players in a straightforward manner, I abstain to fulfill this issue here while the implication is: either one player wins or all players tie, which is hard to exemplify.

Proof. By definition (2.1), $p_i \geq 0$. Also, by definitions (2.1) and (2.2), $\sum_{j \in M} p_j = 1$. Therefore, $\sum_{j \in N} p_j \leq 1$ implies $p_t \geq 0$. But,

$$\sum_{i \in N} p_i \leq 1 \iff \left(\sum_{j \in N} f(x_j) \right)^k \geq \sum_{j \in N} f(x_j)^k$$

Which is true for $k \geq 1$ by Hölder's inequality.⁵ □

Lemma 2.2. *As $k \geq 1$ increases, ties become more likely for any fixed $x = (x_1, x_2) \in \mathbb{R}_+^2$.*

Proof. Let us denote $p_{ri} = \frac{f(x_i)}{\sum_{j \in N} f(x_j)}$, for each $i \in N$. By Lemma 1, $0 \leq p_{ri} \leq 1$, $\forall i \in N$, then:

$$\frac{\partial p_i(x)}{\partial k} = p_{ri}^k \ln p_{ri} \leq 0$$

From definitions (2.1) and (2.2):

$$p_t(x) = 1 - \sum_{i \in N} p_i(x)$$

Therefore,

$$\frac{\partial p_t}{\partial k} = - \sum_{i \in N} p_{ri}^k \ln p_{ri} \geq 0$$

□

Lemma 2.3. *$p_i(x)$ is increasing in own effort and decreasing in any rival's effort.*

Proof. The result follows simply from the partial derivatives of $p_i(x)$ with respect to x_i and x_j , $i \neq j$, respectively:

$$\begin{aligned} \frac{\partial p_i(x)}{\partial x_i} &= \frac{k f'(x_i) f(x_i)^{k-1} \sum_{j \neq i} f(x_j)}{\left(\sum_{l \in N} f(x_l) \right)^{k+1}} \geq 0 \\ \frac{\partial p_i(x)}{\partial x_j} &= - \frac{k f'(x_j) f(x_i)^k}{\left(\sum_{l \in N} f(x_l) \right)^{k+1}} \leq 0 \end{aligned}$$

□

Lemma 2.4. *$p_t(x)$ is increasing in x_i if and only if $x_j > x_i$, and decreasing otherwise.*

⁵See Kuptsov (2001).

Proof. The partial derivative of $p_t(x)$ with respect to x_i is:

$$\frac{\partial p_t}{\partial x_i} = -\frac{k f'(x_i)}{\left(\sum_{l \in N} f(x_l)\right)^{k+1}} \left[f(x_j) \left\{ f(x_i)^{k-1} - f(x_j)^{k-1} \right\} \right] \quad (2.3)$$

Then the following result implies the claim.

$$\text{sign } \frac{\partial p_t}{\partial x_i} = -\text{sign } \left\{ f(x_i)^{k-1} - f(x_j)^{k-1} \right\} \quad (2.4)$$

□

Note that, expression (2.4) is the effective effort difference, scaled by the prevalence of ties, between the players. Each effort difference is scaled by the magnitude of the effective effort of $j \neq i$.

The CSF implies, if $x_2 > x_1$, an increase in the level of effort of player one not only increases player one's probability of winning, by Lemma 2, but also the probability of a tie. On the other hand, an increase the amount of effort of player two leads to an increase in own probability of winning, but decreases the probability of a tie.

Proposition 2.5. $p_t(x)$ achieves a maximum at any $x = (x_1, x_2)$ such that $x_1 = x_2$.

Proof. By Lemma 3, the first derivative of $p_t(x)$ with respect to x_i is zero if and only if $x_i = x_j$. Denote $\hat{x} = (\hat{x}_1, \hat{x}_2)$ such that $\hat{x}_1 = \hat{x}_2$.

The double and the cross derivatives at the same coordinate are, respectively:

$$-\frac{\partial^2 p_t(\hat{x})}{\partial \hat{x}_i^2} = \frac{\partial^2 p_t(\hat{x})}{\partial \hat{x}_i \partial \hat{x}_j} = \frac{k(k-1)f'^2}{2^{k+1}f^2} = A > 0$$

The Hessian, H , at \hat{x} is a 2×2 matrix with entries $h_{ii} = -A$ and $h_{ij} = A$ for $i \neq j$. H has two eigenvalues $\lambda_1 = 0$ and $\lambda_2 = 2A > 0$. Then, H is negative semi-definite, i.e. \hat{x} is either a saddle point or a local maximum. But realize that, by (2.3), $\partial p_t(x)/\partial x_i < 0$ whenever $x_i > x_j$ and positive otherwise, so it cannot be a saddle point. Below we show by contradiction that there is no other set of points maximizing $p_t(x)$ but \hat{x} .

Assume otherwise. Then, $\exists x \in \mathbb{R}_+^2 \cup \{0\}^2$ such that $x_i \neq x_j$ and x satisfies the system of equations which are implied by the necessary FOCs:

$$f(x_j) \left\{ f(x_i)^{k-1} - f(x_j)^{k-1} \right\} = 0$$

Which is impossible. So the only maximal point is \hat{x} .

□

Corollary 2.6. For any $x_1 = x_2$, $p_t(x)$ is maximized and $p_t(x) = 1 - \frac{1}{2^{k-1}}$

The corollary above which follows from Proposition 5 and definition (2.2) states that the probability of the contest ending up in a tie is not only maximized when efforts are equal, but its value also does not depend on the magnitude of the equal efforts spent.

2.3 Equilibrium: existence and uniqueness

We now use the CSF for a non-cooperative game of contest as described in the previous section. Focusing on the symmetric equilibrium, we provide the conditions for its existence and uniqueness.⁶

The setup explained in the previous section implies the following payoff function for each player $i \in N$.

$$u_i = p_i(x) + bp_t(x) - x_i \quad (2.5)$$

which one may rewrite as:

$$u_i = \begin{cases} \left(\frac{f(x_i)}{\sum_{j \in N} f(x_j)} \right)^k + b \left(1 - \frac{\sum_{j \in N} f(x_j)^k}{\left(\sum_{j \in N} f(x_j) \right)^k} \right) - x_i & \text{if } x_j > 0 \text{ for some } j \in N \\ \frac{1}{2^k} + b \left(1 - \frac{1}{2^{k-1}} \right) & \text{if } x_j = 0 \text{ for all } j \in N \end{cases}$$

Proposition 2.7. Let $g(x) = \frac{f'(x)}{f(x)}$. A unique symmetric equilibrium, x^* , exists for the contest with ties iff:

$$b \geq \max \left\{ \frac{\left(g^{-1} \left(\frac{2^{k+1}}{k} \right) 2^k - 1 \right)}{2(2^{k-1} - 1)}, 1 - \frac{k+1}{2(k-1)}, 0 \right\} \quad (2.6)$$

Proof. We abuse the notation as $f_i = f(x_i)$ for brevity whenever necessary. An equilibrium necessarily satisfies the first order conditions derived from (2.5), $\forall i \in N$.

$$\frac{\partial u_i}{\partial x_i} = \frac{k f'_i \left(f_i^{k-1} f_j - b f_j \left(f_i^{k-1} - f_j^{k-1} \right) \right)}{\left(\sum_{l \in N} f_l \right)^{k+1}} - 1 = 0 \quad (2.7)$$

The symmetric solution for (2.7), x_i^* , yields the following for all $i \in N$.

$$g(x_i^*) = \frac{f'(x_i^*)}{f(x_i^*)} = \frac{2^{k+1}}{k} \quad (2.8)$$

⁶In Appendix B we present the results for asymmetries in terms of cost functions.

First, $g(x) = \frac{f'(x)}{f(x)}$ is unbounded from above, $\lim_{x \rightarrow \infty} g(x) = 0$, and strictly decreasing by the strict concavity of $f(x)$. As $\frac{2^{k+1}}{k}$ is a positive scalar, there is a unique x_i^* satisfying (2.8). Thus, there can only be a single symmetric equilibrium.

By equations (2.1) and (2.2), the solution x^* implies the following $\forall i \in N$:

$$p_i^* = \frac{1}{2^k} \quad \text{and} \quad p_t^* = 1 - \frac{1}{2^{k-1}} \quad (2.9)$$

$$u_i^* = u^* = \frac{1 + b(2^{k-1})}{2^k} - g^{-1}\left(\frac{2^{k+1}}{k}\right) \quad (2.10)$$

An equilibrium necessarily satisfies the second order conditions (SOC) locally of each player.⁷

$$\frac{\partial^2 u^2(x^*)}{\partial x_i^{*2}} = \frac{k}{2^{k+2}} \left\{ \frac{f''}{f} + \left(\frac{f'}{f}\right)^2 [2(k-1)(1-b) - (k+1)] \right\} \leq 0 \quad (2.11)$$

Then a sufficient condition for the SOC to be locally satisfied is:

$$b \geq \max \left\{ 1 - \frac{k+1}{2(k-1)}, 0 \right\} \quad (2.12)$$

Last, for x^* to be an equilibrium it is necessary that $u(x_i^*, x_j^*) \geq u(0, x_j^*)$. Then by (2.10) we get:

$$b \geq \max \left\{ \frac{\left(g^{-1}\left(\frac{2^{k+1}}{k}\right) 2^k - 1\right)}{2(2^{k-1} - 1)}, 0 \right\} \quad (2.13)$$

□

Some remarks about the existence are in order. First, as $b \geq 0$, the sufficiency (2.12) is satisfied for any $b \in [0, 1]$ if $k \leq 3$. Note also that there is always a tie-prize $b \in [0, 1]$ such that (2.12) holds, as $\frac{k+1}{2(k-1)}$ is always positive.

We cannot provide a sufficient condition for (2.13) without an explicit form for f . To exemplify, let $f(x) = x^\alpha$, where $\alpha < 1$ by strict concavity. Then,

$$b \geq \max \left\{ \frac{\alpha k - 2}{4(2^{k-1} - 1)}, 0 \right\} \quad (2.14)$$

Condition (2.14) holds for any $b \in [0, 1]$ whenever $k \leq 2$. Also, as in condition (2.12), there is always a tie-prize fulfilling the condition (2.14).

Also, it is natural to ask how strict this conditions is. In order to provide an example take $f(x) = x^{0.9}$, and $k = 5$. Then the symmetric equilibrium would not exist if $b < 0.25$.

⁷The sufficiency of local SOC depends on the particular form of function f . But, in Appendix B we show the global SOC admits the sufficiency of local SOC whenever $f(x) = x^\alpha$ and $\alpha \leq 1$.

Proposition 2.8. *The symmetric equilibrium for the contest with ties is the unique equilibrium if $k \leq 2$.*

Proof. In the equilibrium, the marginal payoff of the two contestants are equal. Thus, using (2.7):

$$f_i^{k-1} f_j^{k-1} \left(\frac{f'_i}{f_j^{k-2}} - \frac{f'_j}{f_i^{k-2}} \right) = b \left(f_i^{k-1} - f_j^{k-1} \right) (f'_i f_j + f'_j f_i) \quad (2.15)$$

For $x_i = x_j$, (2.15) is obviously satisfied. Now assume wlog $x_i > x_j$, then right hand side is positive. By concavity of f , $f'_i < f'_j$ and if $k < 2$, $f_j^{k-2} > f_i^{k-2}$. Then the left hand side is negative, which contradicts the equality. \square

2.4 Equilibrium properties

In this section, I assume that the conditions on existence are satisfied and study the properties of the symmetric equilibrium given by equation (2.8). In particular, I compare the equilibrium effort for the contest with ties, i.e. $k > 1$, and the contest without ties, i.e. $k = 1$. For analytical purposes, I assume $f(x) = x^\alpha$ such that $\alpha < 1$ throughout.

By (2.8), the equilibrium effort when ties are allowed and when they are not is, respectively:

$$x^* = \frac{\alpha k}{2^{k+1}}, \quad x_c^* = \frac{\alpha}{4} \quad (2.16)$$

Proposition 2.9. *In equilibrium, players' equilibrium efforts are larger when ties are possible if and only if $k \in (1, 2)$.*

Proof.

$$x^* \geq x_c^* \iff k^{\frac{1}{k-1}} \geq 2 \quad (2.17)$$

$k^{\frac{1}{k-1}}$ strictly decreasing, $\lim_{k \rightarrow 1} k^{1/k-1} = e$, and $\lim_{k \rightarrow \infty} k^{1/k-1} = 1$. Thus, there is a unique solution to $k^{\frac{1}{k-1}} = 2$, which is $k = 2$. Thus, $k < 2$ implies $k^{\frac{1}{k-1}} > 2$, otherwise $k^{\frac{1}{k-1}} \leq 2$ \square

By (2.8), it is clear that the equilibrium effort does not depend on the value of the tie-prize. Moreover, as stated in the previous section, there may still be an equilibrium even when $b = 0$ in which players spend more in equilibrium by Proposition 4. This result is somehow surprising, as when $b = 0$ the payoff function (2.5) is strictly less than the one without ties for any effort level. So this increase in efforts cannot be explained by the mitigating effect of having an extra tie-prize. We show below that this result is emanating from the change in the shape of best-response of functions once ties are allowed.

Lemma 2.10. Denote the best response strategy of player i to any strategy x_j as $x_{ic}^*(x_j)$ when there is no tie, and $x_{it}^*(x_j)$ when there is. Then, given $b = 0$:

$$x_{it}^*(x_j) \geq x_{ic}^*(x_j) \Leftrightarrow f(x_{it}^*(x_j)) \left(k^{\frac{1}{k-1}} - 1 \right) \geq f(x_j) \quad (2.18)$$

Proof. Denote $f(x_j) = y$. The best response to y , for the two cases are defined implicitly by the FOC:

$$k \left(\frac{f(x_{it}^*(x_j))}{f(x_{it}^*(x_j)) + y} \right)^{k-1} \frac{f'(x_{it}^*(x_j)) y}{(f(x_{it}^*(x_j)) + y)^2} = 1 = \frac{f'(x_{ic}^*(x_j)) y}{(f(x_{ic}^*(x_j)) + y)^2}$$

Manipulating the expression we get:

$$k \left(\frac{f(x_{it}^*(x_j))}{f(x_{it}^*(x_j)) + y} \right)^{k-1} = \frac{f'(x_{ic}^*(x_j))}{f'(x_{it}^*(x_j))} \left(\frac{f(x_{it}^*(x_j)) + y}{f(x_{ic}^*(x_j)) + y} \right)^2 \quad (2.19)$$

If $x_{it}^*(x_j) > x_{ic}^*(x_j)$ then, the right hand side of (2.19) is larger than 1 by the concavity of f . Then,

$$k \left(\frac{f(x_{it}^*(x_j))}{f(x_{it}^*(x_j)) + y} \right)^{k-1} > 1$$

Which implies the result. \square

Lemma 2.11. $x_i^*(x_j) \geq x_j$ if and only if $x_j \leq x_{jt}^*$, where x_{jt}^* is the equilibrium effort of player $j \neq i$.

Proof. Let the first order condition be denoted as:

$$R(k, x) = k \left(\frac{f_i}{f_i + f_j} \right)^{k-1} \frac{f'_i f'_j}{(f_i + f_j)^2} - 1$$

$\frac{dx_{it}^*(x_j)}{dx_j} = -\frac{R_{x_j}}{R_{x_i}}$. Denote the SOC by $S < 0$ and $\kappa = k f_i^{k-1} \frac{f'_i f'_j}{(f_i + f_j)^{k+2}}$. Then,

$$\frac{dx_{it}^*(x_j)}{dx_j} = \frac{\kappa}{-S} [(f_i - f_j) - (k-1) f_j]$$

At the symmetric equilibrium this expression is negative. Also for $x_j = 0$, $x_i^*(x_j) > 0$ otherwise $x = (0, 0)$ would be an equilibrium. Then, the difference $\Delta x = x_i^*(x_j) - x_j > 0$ for $x_j < x_{jt}^*$, by the uniqueness of the symmetric equilibrium for $k \leq 2$. \square

Lemma 2.12. $x_{it}^*(x_j) > x_{ic}^*(x_j)$ for all $x_j < x_{jt}^*$ if and only if $k < 2$.

Proof. Lemma 5 show $x_{it}^*(x_j)$ should be sufficiently large for a given x_j for this best response to be larger than the one when ties are not allowed. Lemma 6 shows that this best response is indeed sufficiently large in the domain $x_j \in [0, x_{jt}^*]$ for $k < 2$. \square

The proposition above explains where the larger equilibrium efforts when ties are possible follows from. Figure 2.1 below presents the result: as long as the rival strategy is sufficiently small, the best response to that strategy with ties is larger. When $k \leq 2$, this threshold line lies below $x_2 = x_1$, therefore contest with ties achieves a larger equilibrium effort.

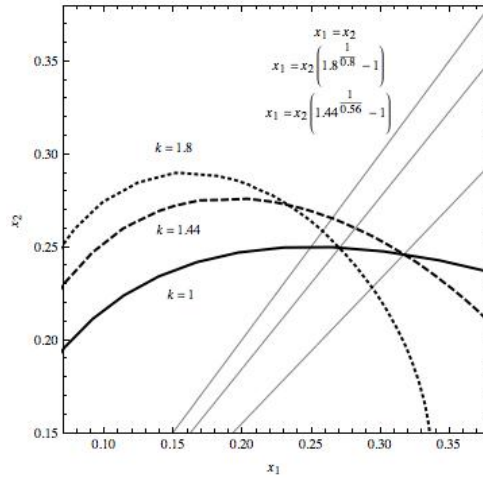


Figure 2.1: Best-response functions when $b = 0$ ⁸

We already know that tie prize does not play any role in equilibrium payoffs. However, in the next lemma we show it effects the best-responses profoundly.

Proposition 2.13. Assume $k \leq 2$, then for $b > 0$ $\frac{\partial x_i^*}{\partial b} < 0$ if $x_i > x_j$, otherwise $\frac{\partial x_i^*}{\partial b} > 0$.

Proof. Define $\bar{R}(k, x, b) = R(k, x) + b\lambda(f_j^{k-1} - f_i^{k-1})$. This is the FOC when $b > 0$, where $\lambda = \frac{k f_j f_i'}{(f_i + f_j)^{k+1}} > 0$. Then

$$\frac{dx_i^*(x_j)}{db} = -\frac{\bar{R}_b}{\bar{R}_x} = \frac{\lambda(f_j^{k-1} - f_i^{k-1})}{-S}$$

As $S < 0$ by definition, it is clear that the numerator is negative if $x_i > x_j$, and positive otherwise. \square

As Figure (2.2) shows, the result we obtain implies that given a strategy of the rival, an increase in the tie-prize will decrease the best response to that strategy if the best-response is already larger than the rival strategy which is the case if rival strategy is less than the equilibrium value by Lemma 6. It will increase the best-response otherwise. We also see that the best-response functions intersect at the same coordinate implying that the equilibrium does not depend on the tie-prize.

⁸The equations from top to bottom correspond to the lines from northwest to southeast, respectively.

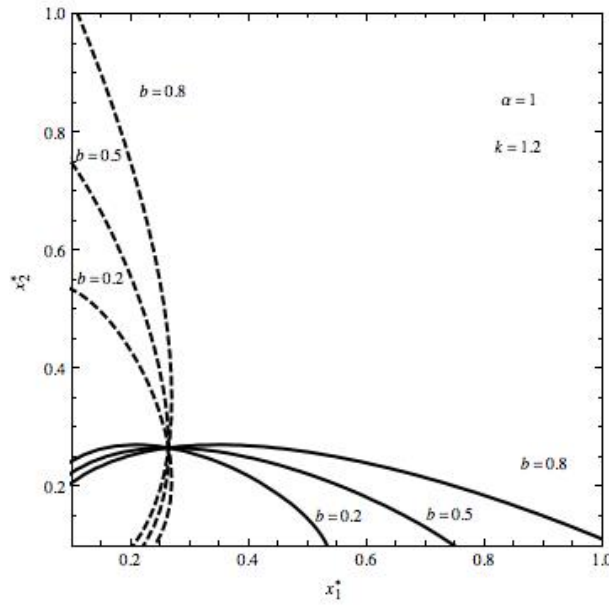


Figure 2.2: Best response functions with respect to different tie-prizes.

Obviously, the result above does not play any role in the current symmetric setting. However, as we show below, it has an important bite when the players are asymmetric in terms of the resources they can spend for the contest.

Theorem 2.14. *If player $j \neq i$ is constrained in resources, i.e. $x_j^*(x_i) < x_{jt}^*$ for all $x_i \in \mathbb{R}_+$, total equilibrium effort decreases if tie-prize increases and decreases otherwise.*

Proof. Assume player j is constrained by $\hat{x}_j < x_j^*$, then the best response of player j is defined by

$$x_j^*(x_i) = \begin{cases} x_j^*(x_i) & \text{if } R(k, b, \hat{x}_j, x_i) < 1 \\ \hat{x}_j & \text{otherwise} \end{cases} \quad (2.20)$$

By Proposition 5, $x_i^*(x_j^*(x_i)) > x_j^*(x_i)$. Thus, in equilibrium $x_{it}^* > x_{jt}^* = \hat{x}_j$. By lemma 7, $\frac{dx_i^*}{db} < 0$ and $\frac{dx_j^*}{db} > 0$. However as j is constrained by \hat{x}_j , the best response is the same. Then the total investment will fall due to an increase in the tie prize. \square

It is also noteworthy that, as k increases its effect on the symmetric equilibrium is not necessarily monotonic. We present the result in the following proposition.

Proposition 2.15. *Equilibrium effort is non monotonic in k when ties are allowed and has a unique maximum at $k = \frac{1}{\ln 2}$.*

Proof.

$$\frac{\partial x^*}{\partial k} = \frac{\alpha}{2^{k+1}} (1 - k \ln 2)$$

As $k \geq 1$, for values of k satisfying $k < 1/\ln 2 \simeq 1.44$ the equilibrium effort is strictly increasing. Otherwise it is decreasing. \square

Proposition 6 also implies that the difference between the contest with ties and the contest without ties is maximum when $k = 1/\ln 2 \simeq 1.44$. This result is important in the sense that if a contest designer wishes to maximize total effort in a two-player symmetric contest, the optimal tie-proneness is $k \simeq 1.44$.

2.5 Empirical application

2.5.1 Data and estimation method

In order to assess the performance of the CSF empirically, accompanied by the success functions of [Jia \(2012\)](#) and [Blavatsky \(2010\)](#), I use the data for four major soccer leagues.⁹ It contains observations for each league match played from season 2004-2005 to 2012-2013, first half. The leagues included are Spanish *Primera División*, German *Bundesliga*, Italian *Serie A*, and French *Ligue 1*. Each observation contains the market value of the line up of each team, the information identifying the home and the visiting team and the result of the match.

I use market values of the line up as a proxy of the efforts put by the club as the decision maker and employ a maximum likelihood estimation, as in [Greene \(2003\)](#), to obtain the estimates and inference based on them. By using the maximum likelihood estimator we implicitly assume that the real data generating process is given by a CSF. The log-likelihood function to be maximized is:

$$\ell(y|x, \Phi) = \sum_{n=1}^N \sum_{y_n \in \{0,1,2\}} \sum_{h_n \in \{1,2\}} \mathbb{1}(y_n) \mathbb{1}(h_n) \log(P(y_n|x_n, h_n, \Phi)) \quad (2.21)$$

where Φ is the model specific vector of parameters, $x = (x_1, x_2)^N$ is market values of the two teams for each match, $h = \{1, 2\}^N$ is the information on home and visiting team, $y \in \{0, 1, 2\}^N$ is the results of every match.

x_{in} is the market value of the line-up of the team i for the match n , in thousand pounds. $h_n \in \{1, 2\}$ shows which team plays in the home field. $y_n = 1$ and $y_n = 2$ implies that there is a winner of the match and $y_n = 0$ shows that the match ends in a tie. Denoting $i, j \in \{1, 2\}$,

⁹Data is acquired from *transfermarkt.co.uk*.

$i \neq j$ as the team labels, $P(y_n|x_n, h_n, \Phi)$ is the following pointwise likelihood function:

$$P(y_n|x_n, h_n, \Phi) = \begin{cases} \pi_{iin} & \text{if } y_n = h_n = i \\ \pi_{jin} & \text{if } y_n = j, h_n = i \\ \pi_{tin} = 1 - \pi_{iin} - \pi_{jin} & \text{if } y_n = 0, h_n = i \end{cases} \quad (2.22)$$

π_{iin} is the probability of player i winning the match n when he is the home team, hence $y_n = h_n$, π_{jin} is the probability that player j winning the match n when he is the visiting team. π_{tin} is the remaining probability, i.e. the probability of a tie.

The explicit form of the pointwise likelihood function is determined by the particular CSF used. I use four CSFs that allow for ties for two different specifications: excluding and including the effect of playing at home field. Below I show the CSFs I use explicitly.

$$\pi_{iin}^B(x_n) = \frac{\varphi x_{in}^\alpha}{a + \varphi x_{in}^\alpha + x_{jn}^\alpha}, \quad \pi_{jin}^B(x_n) = \frac{x_{jn}^\alpha}{a + \varphi x_{in}^\alpha + x_{jn}^\alpha} \quad (2.23)$$

$$\pi_{iin}^J(x_n) = \frac{\varphi x_{in}^\alpha}{\varphi x_{in}^\alpha + c x_{jn}^\alpha}, \quad \pi_{jin}^J(x_n) = \frac{x_{jn}^\alpha}{\varphi c x_{in}^\alpha + x_{jn}^\alpha} \quad (2.24)$$

$$\pi_{iin}^A(x_n) = \left(\frac{\varphi x_{in}}{\varphi x_{in} + x_{jn}} \right)^k, \quad \pi_{jin}^A(x_n) = \left(\frac{x_{jn}}{\varphi x_{in} + x_{jn}} \right)^k \quad (2.25)$$

$$\pi_{ii}^N(x) = \frac{1 + \varphi}{3}, \quad \pi_{ji}^N(x) = \frac{1 - \varphi}{3} \quad (2.26)$$

The first CSF is due to [Blavatsky \(2010\)](#) (henceforth BSF) where $a > 0$. The second is due to [Jia \(2012\)](#) (henceforth JSF) where $c > 1$. The third is the CSF analyzed in this paper (henceforth ASF) where $k > 1$. The last specification is the benchmark random choice (naive) model (henceforth, NF).¹⁰ The parameter $\varphi > 0$ signifies the home (dis)advantage. When there is no home advantage or disadvantage present $\varphi = 1$.¹¹

We use two functional forms for $f(x)$. The first is the Tullock's effective effort function, $f(x) = x^\alpha$. The other is the logarithmic function $f(x) = \log(1 + x)$.

For the comparison within different specifications of a CSF, I use the likelihood ratio test. However comparison of different CSFs requires a different approach as these models are non-nested. Therefore, I use the [Vuong \(1989\)](#) closeness test for non-nested models. I present the estimation results and the results from the Vuong's test for each league in the Tables [B.1](#), [B.2](#), and [B.3](#), in Appendix B.

¹⁰Note that the parameter φ in the Naive model is not equivalent to the same parameter in the other specifications. We preferred to abuse the notation in order not to cause a confusion in the presentation of the estimation results.

¹¹ $\varphi = 0$ for the Naive model.

2.5.2 Discussion

First of all, the results indicate that the parameters estimated for each CSF are significant. The functional form $f(x) = x^\alpha$ fits better than the logarithmic function for every league and the likelihood ratio test indicates that the addition of the home effect, which turns out to be an advantage as expected, significantly improves the model.

Comparison of CSFs reveal that JSF and ASF turn out to be equally appropriate. In particular, the former performs better for Spanish and German leagues and the latter performs better for Italian and French leagues. However, these differences in performance are not significant. Thus, one cannot confirm a general favoring of one over the other.

Moreover, JSF and ASF always perform better than the BSF. This tendency is highly significant for the Italian league, but for the other three competitions one can only confirm the superiority in 10% significance level for one-tailed test. This result partially confirms the observation of [Peeters and Szymanski \(2012\)](#) who discuss the implausibility of BSF for sports. The results suggest also that there is a tendency for the outcome of the match being prone to the difference between the investment levels of the two competing teams no matter how large the investments of those two teams are.

Lastly, all CSFs perform better than the NF. Even though the log-likelihood of the NF improves with the addition of the home effect the improvement is surpassed by any of the CSF I use.

For the comparison between leagues I use ASF and JSF because of their comparatively superior performance. I also further refine the interpretations by using the functional form $f(x) = x^\alpha$ and consider the estimations using home effect due to their conclusive superiority.

First of all, the estimated values for tie-proneness, k , and coarseness, c , are in conformity with the particular characteristics of the different leagues as I present in [Table 2.1](#).

Data shows that the competition with the highest frequency of ties is the French league. As expected, the largest tie-proneness and coarseness is for that competition. Using ASF, it is observed that the estimated tie-proneness is very close to the value where the efforts are maximized, i.e. $k \simeq 1.44$.¹²

I used the Gini coefficient¹³ and the mean total investment to see if there is a relation with the frequency of ties and the discrepancy of investment or the total investment. Data shows that the least frequency of the ties is in the Spanish league which has the largest mean total investment and Gini coefficient. Figures 3 to 6 below also confirm the discrepancy in the investments

¹²Even though this maximized value is from the symmetric CSF, the estimated values without the home effect also confirms the result.

¹³Gini coefficient in our framework designates the level of discrepancy in market values across teams. Thus, a larger Gini coefficient points out a larger inequality across the teams in a league in terms of the market values of the players.

Table 2.1: Tie Statistics

League	\hat{k}	\hat{c}	Tie Frequency	MTI	Gini
Spanish	1.426 (0.025)	1.721 (0.050)	0.22	5717.45	0.59
Italian	1.544 (0.030)	1.964 (0.066)	0.28	5208.77	0.48
German	1.482 (0.024)	1.826 (0.048)	0.26	4230.21	0.32
French	1.562 (0.026)	1.981 (0.055)	0.30	3129.53	0.32

Note: This table reports the estimation results of ASF and JSF including the summary statistics concerning ties and market values: \hat{k} and \hat{c} are the estimated values of tie-proneness and coarseness parameters, respectively. Tie frequency is the percentage of ties in all the matches played. Mean Total Investment (MTI) is the mean market values of teams covering all the seasons and Gini coefficient is calculated using the share of market values for each team.

in Spanish league and show that French league is rather symmetric in terms of investments compared to the other leagues included in the data. Moreover, the lowest levels of mean total investment and Gini coefficient also coincides with the highest tie frequency which is for the French league. However, to confirm a general tendency requires data from other leagues. Thus, I leave this issue as a further research interest.

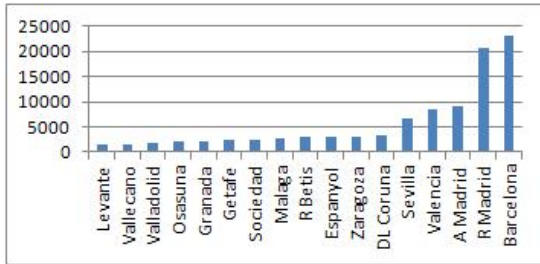


Figure 2.3: Spain

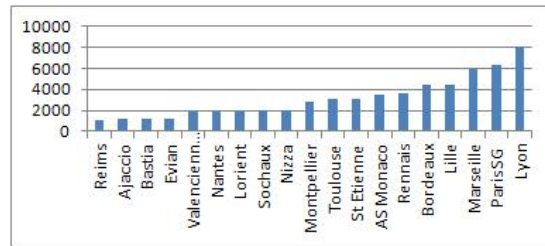


Figure 2.4: France

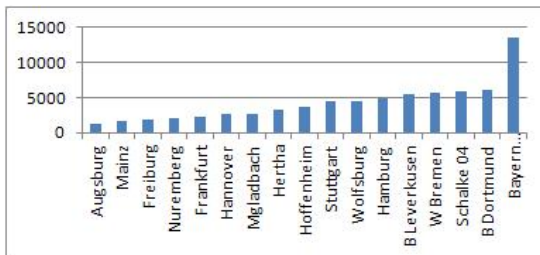


Figure 2.5: Germany

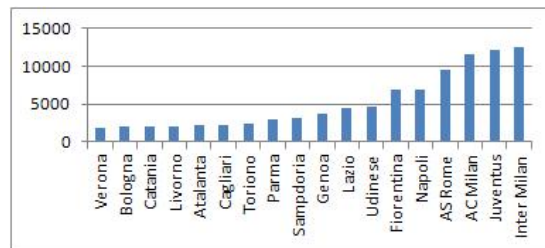


Figure 2.6: Italy

I would also like to mention briefly the estimated parameters that do not directly influence a tie as presented in Table 2.2. These parameters are the ‘decisiveness’, α , and the ‘home-advantage’, φ , parameters.

Table 2.2: Home effect and the decisiveness parameters

League	$\hat{\alpha}^A, \hat{\alpha}^J$	$\hat{\varphi}^A, \hat{\varphi}^J$
Spanish	0.486, 0.550	1.540, 1.610
Italian	0.560, 0.647	1.599, 1.719
German	0.611, 0.701	1.425, 1.505
French	0.393, 0.454	1.539, 1.655

Note: This table reports the estimation results of ASF and JSF for the home effect and the decisiveness parameter: $\hat{\alpha}^A$ and $\hat{\alpha}^J$ are the estimated values of decisiveness parameter; and $\hat{\varphi}^A$ and $\hat{\varphi}^J$ are the estimated values for the home effect, respectively for ASF and JSF.

Estimated values of the parameters mentioned point out that the investments affect the outcome the most in German league and the least in the French league. Moreover, the home advantage is the most dominant in the Italian league and the least dominant in the German league. ASF and JSF agree on all the results.

2.6 Conclusions

Ties are a natural outcome of a contest or a tournament in various contexts. I suggest a new CSF that allows for ties, I analyze the game implied by this function and the related contest, and I empirically assess the performance of the CSF using data from soccer.

The CSF I suggest possesses intuitive properties regarding ties. First of all, it reaches a maximum value when the efforts of the contestants are identical, and this value is the same no matter how large these identical efforts are. Moreover, increasing the effort of a player that has a humble investment in the contest increases the likelihood of a tie, and decreasing it reduces this likelihood. Our CSF also preserves the common axioms demanded from a CSF such as positive response to probability of winning possessed by a player to an increase in own effort, and negative response to it with respect to an increase in rival effort.

Our analysis of the Nash equilibrium of the game implied shows that equilibrium efforts do not depend on the prize the contestants get should the game results in a draw. Moreover, for the two player case, equilibrium efforts are larger compared to the contest with no ties given that the contest is not very prone to ties. When there are more than two players, because of the opportunity of the free-riding in achieving a tie, the efforts are always less than the contest without the possibility of a tie.

In the empirical estimation of our CSF, by adding a parameter of home advantage, I find that the two CSFs, our proposed and the one by Jia (2012), that imply the importance of the difference

in efforts being decisive on ties, perform the best. The estimation results also point out that home advantage is a major determinant of the outcome. For example, the estimated parameters for the Spanish *Primera División* point out that, in a case of contest involving two teams with equal investments, home team has approximately 0.490 chances of winning, while visiting team have 0.265, and a draw has 0.245 probability of taking place.

Chapter 3

Fighting for bargaining power

3.1 Introduction

“War is thus an act of force to compel our enemy to do our will.”

*Carl von Clausewitz.*¹

The quest to find a fulfilling answer to the question of why armed conflicts take place has a long history. Earlier works, except [Machiavelli \(1961\)](#), rely heavily on irrational behavior and misperceptions. The pioneering works of [Intriligator and Brito \(1984\)](#) and [Fearon \(1995a\)](#) are considered as the first formal approaches to point out that armed conflict could emerge as an equilibrium in a game played by rational players. [Intriligator and Brito \(1984\)](#) stresses the instability of peace when adversaries move from states of total disarmament to arms race, or vice versa. Whereas [Fearon \(1995a\)](#) pinpoints that in the presence of a costless bargaining option, the only possible way for an open conflict is asymmetric information. In their static setting with symmetric information [Anbarci, Skaperdas, and Syropoulos \(2002\)](#) implicitly demonstrate that costless bargaining in the shadow of conflict - or armed peace - is Pareto dominant to open war.

These results invoked two different approaches to the problem. In one approach with a partial equilibrium setting in which bargaining options are severed by the inability to commit, [Beviá and Corchón \(2010\)](#) show that it is possible to sustain peace with resources transferred from the richer adversary to the poorer one. In the other approach initiated by [Garfinkel and Skaperdas \(2000\)](#) and proceeded by [McBride and Skaperdas \(2006\)](#), armed conflict may dominate peaceful solutions with armed peace when there is a long shadow of future, i.e. when an armed conflict alters the strategic positions of each adversary in a different way a peaceful resolution does. In the aforementioned papers, long term contracts for arming cannot be enforced and the victor

¹[Clausewitz \(1984\)](#).

of an armed conflict establishes secure property rights on the resource/territory contested. In that manner, cooperation is deeply severed due to future gains of an armed conflict. While the former paper abstract the future gains of war in a compact resource attainable by the victor in post-war period, the latter transfers this approach to a dynamic setting, in which a victory is accomplished in a series of consequent battles. In a related paper, [Bester and Konrad \(2005\)](#) identify the timing of the initiation of an armed conflict in a dynamic setting where there is a long shadow of future. A dynamic resource war is also modeled by [Acemoglu, Golosov, Tsyvinski, and Yared \(2012\)](#). In their framework a resource-rich country which produces a marketable good is dynamically threatened by a resource-poor country which can declare and capture part of a resource instead of purchasing at market price. They find that the elasticity of demand of this good in resource-poor country determines whether peace can be sustained or not. In a related paper [Yared \(2010\)](#) considers a dynamic game of resource transfer between an aggressive and non-aggressive country in which the aggressive country may declare war in case no transfer occurs from the non-aggressive country. However, the aggressive country is uncertain whether the failed transfers are due to lack of cooperation or high costs of transfers. A sequential equilibrium of temporary war along the equilibrium may exist if countries are sufficiently patient, the costs of war is high and the cost of transfers for the non-aggressive country is small. If these conditions are not met, the only sequential equilibrium that may emerge is total war.

In many instances of conflict, including armed conflict and many others such as pack leadership struggle in the nature, one adversary necessarily needs to damage his contestant in order to ascertain a victory via the surrender or the total destruction of the forces and/or the strength of his adversary. This observation sheds light on the importance of intra-conflict strategic positions of each adversary and invokes the importance of damaging power of the forces in a short time horizon of an ongoing conflict. The damaging power can also be interpreted as sabotage in the manner described by [Konrad \(2000\)](#) in which players decide on the harmful and productive effort, respectively.

In this paper I introduce a two-player dynamic war where victory is not achieved until one side is totally eradicated or inclined to surrender and the other adversary is willing to stop fighting. Players differ in their valuations of the resource. In an initial stage, each player decides on the costly forces he would possess. After the initial stage is over, they decide whether to fight or remain peaceful. War initiates if at least one player chooses to fight, and in that case the armed conflict transforms into an indefinite horizon of series of battles. If not, players share the resource according to the bargaining power they possess. Given that war starts, in each stage of the conflict, players decide whether to stop fighting or continue. If at least one player chooses to continue, they damage adversary forces as a function of their relative military standing. After one stage of conflict is over, a new bargaining position over the contested resource arises while the remaining forces of each side dictates a new relative military strength. The conflict is terminated

when both players decide to stop fighting and they share the resource at the current bargaining power they possess in that case.

I find, in the subgame that starts after the initial stage is over, that the ratio of forces decided in the initial stage, the efficacy of armed forces in destruction and bargaining, and the fraction of future payoffs discounted by adversaries completely characterize the length of war. War does not start if the players discount future payoffs heavily or when the relative amount of forces are either very small or very large. On the other hand, until a threshold value is reached, the length of war decreases with the relative strength of one adversary. Moreover, if the effectiveness of forces in bargaining power is large, war lasts shorter.

I also find that, in the subgame perfect equilibrium, the initial forces are such that the outcome is most likely an armed peace.

The rest of the paper goes as follows: in Section 2 I present the model, in Section 3 we solve for the equilibrium in the subgame that starts after the initial stage of arming, in Section 4, we find the subgame perfect Nash equilibrium of the game, and in Section 5 I conclude.

3.2 Model

Two players $i = 1, 2$ are competing for a continuously divisible resource, which they value as v_1 and v_2 , respectively. Without loss of generality I assume $v_1 > v_2$. The competition is in the form of a dynamic conflict of indefinite horizon with an initial stage of effort decisions. At the initial stage each player simultaneously chooses the amount of forces, x_i . The forces are costly and I assume $c_i(x_i) = x_i$ for $i = 1, 2$. I also assume that the investment on forces is not redeemable.² After these choices are made, they are observed by each player. For further reference, I will denote the relative initial stage arming as $r = \frac{x_2}{x_1}$.

After the initial choice is over, each player simultaneously chooses an action from the set $A = \{S, F\}$, where F is to fight and S is to stop fighting, at every $t \in \{0, 1, 2, \dots\}$. If at least one player chooses F at a given period t , conflict takes place and each player reduces his rival forces with respect to a destruction function $D_i(x_1^t, x_2^t)$, where x_t is the remaining forces at t , obtain zero payoff at t , and in the next period they achieve a bargaining power $B_i(x_1^{t+1}, x_2^{t+1})$ and a destructive power $D_i(x_1^{t+1}, x_2^{t+1})$. If both choose S at period t , conflict ends and they gain $B_i(x_1^t, x_2^t) v_i$.

The bargaining power may be interpreted as the decision rule of an external individual such as an arbitrator, peace organization or the public opinion. I assume the following particular form

²This assumption reflects that the technology investment, training and transfers necessary to possess an armed force such as majority of the components of spending for an army such as military bases, various weapons and wages cannot be recuperated as it would be expected at least for warfare before 20th century.

for the bargaining power.

Definition 3.1. Let x_1^t and x_2^t represent the forces of player one and player two, after $t - 1$ periods of armed conflict, respectively; and $\beta > 0$. The bargaining power each player possesses at t is:

$$B_1(x_1^t, x_2^t) = B(x_1^t, x_2^t) = \begin{cases} \frac{(x_1^t)^\beta}{(x_1^t)^\beta + (x_2^t)^\beta} & \text{if } \max\{x_1^t, x_2^t\} > 0 \\ \frac{1}{2} & \text{if } x_1^t = x_2^t = 0 \end{cases} \quad (3.1)$$

and $B_2(x_1^t, x_2^t) = 1 - B(x_1^t, x_2^t)$.

The bargaining power I use is motivated primarily by [Anbarci, Skaperdas, and Syropoulos \(2002\)](#) as a bargaining solution in the shadow of conflict.

The destructive power of player j determines the amount of reduction in the forces possessed by player i . Thus, after the conflict in period t , the forces player i possesses at $t + 1$ is reduced to:

$$x_i^{t+1} = x_i^t (1 - D_j(x_i^t, x_j^t)) \quad (3.2)$$

The destructive power represents the damaging power of an army or alike. In an armed conflict the number of the soldiers, firearms, etc. determine the capacity of that army to reduce the rival forces. Motivated from this observation I assume the following form for the destructive power.

Definition 3.2. Let x_1^t and x_2^t represent the forces of player one and player two, after $t - 1$ periods of armed conflict, respectively; and $\alpha > 0$. The destructive power each player possesses at t is:

$$D_1(x_1^t, x_2^t) = D(x_1^t, x_2^t) = \begin{cases} \frac{(x_1^t)^\alpha}{(x_1^t)^\alpha + (x_2^t)^\alpha} & \text{if } \max\{x_1^t, x_2^t\} > 0 \\ 0 & \text{if } x_1^t = x_2^t = 0 \end{cases} \quad (3.3)$$

and $D_2(x_1^t, x_2^t) = 1 - D(x_1^t, x_2^t)$.

A history, $(x, a^0, \dots, a^{t-1}) = h^t \in \mathcal{H}^t$ for this game is a sequence of actions until time t , where $x = (x_i, x_j) \in \mathbb{R}_+^2$ are the initial force allocations and $a^t \in \{F, P\}^2$ for all $t \in \{0, 1, 2, \dots\}$ and \mathcal{H}^t is the set of all possible histories at stage t . A strategy, s_i , for player $i \in \{1, 2\}$ is a sequence of maps $s_i^t : \mathcal{H}^t \rightarrow A$ where $A = \mathbb{R}_+^2$ or $A = \{F, P\}$.

Each player discounts future payoffs at rate $\delta \in (0, 1)$. Then the gain from the conflict assuming that it stops at time t is:

$$v_i g_i(x_1^t, x_2^t) - x_i = v_i \delta^t B_i(x_1^t, x_2^t) - x_i \quad (3.4)$$

In the next section I will begin by solving the subgame that starts after initial stage decisions. I use subgame perfection to solve this game. Therefore, the equilibrium achieved in this subgame has to be a part of the equilibrium strategy for the whole game of conflict. For simplicity, I will focus on the pure strategy equilibria.

3.3 Equilibrium of the subgame

I begin this section by studying how bargaining power and the forces possessed by each player evolve over time given the initial choice of forces. The current state of these two variables will be the factor determining the decision to fight or to stop fighting at each stage, t . The following lemma summarizes the rule they obey as a function of the initial forces possessed.

Lemma 3.3. *Given that $\max\{x_1, x_2\} > 0$, and the conflict continues for $t - 1$ stages, the remaining forces, bargaining power and destructive power of $i \in \{1, 2\}$ at stage t are respectively given as follows:*

$$x_i^t = \frac{(x_i)^{(1+\alpha)^t}}{\prod_{\tau=0}^{t-1} \left(\sum_{k=1}^2 (x_k)^{(1+\alpha)^\tau \alpha} \right)} \quad (3.5)$$

$$B_i^t = \frac{(x_i)^{(1+\alpha)^t \beta}}{(x_1)^{(1+\alpha)^t \beta} + (x_2)^{(1+\alpha)^t \beta}} \quad (3.6)$$

$$D_i^t = \frac{(x_i)^{(1+\alpha)^t \alpha}}{(x_1)^{(1+\alpha)^t \alpha} + (x_2)^{(1+\alpha)^t \alpha}} \quad (3.7)$$

Proof. The result is attained using induction. Assume equations (3.5) and (3.6) are valid, and that the conflict continues at $t + 1$. Then by equation (3.2)

$$\begin{aligned} x_i^{t+1} &= \frac{(x_i)^{(1+\alpha)^t}}{\prod_{\tau=0}^{t-1} \left(\sum_{k=1}^2 (x_k)^{(1+\alpha)^\tau \alpha} \right)} \left(1 - \frac{(x_i)^{(1+\alpha)^t \alpha}}{(x_1)^{(1+\alpha)^t \alpha} + (x_2)^{(1+\alpha)^t \alpha}} \right) \\ &= \frac{(x_i)^{(1+\alpha)^{t+1}}}{\prod_{\tau=0}^t \left(\sum_{k=1}^2 (x_k)^{(1+\alpha)^\tau \alpha} \right)} \end{aligned}$$

Then, by definitions of B_i^t and D_i^t :

$$\begin{aligned}
B_i^{t+1} &= \frac{(x_i)^{(1+\alpha)^{t+1}\beta}}{(x_1)^{(1+\alpha)^{t+1}\beta} + (x_2)^{(1+\alpha)^{t+1}\beta}} \\
D_i^{t+1} &= \frac{(x_i)^{(1+\alpha)^{t+1}\alpha}}{(x_1)^{(1+\alpha)^{t+1}\alpha} + (x_2)^{(1+\alpha)^{t+1}\alpha}}
\end{aligned} \tag{3.8}$$

Since t and $t + 1$ are arbitrary, equations (3.5) and (3.6) are valid. \square

Lemma 3.4. *Given $\max\{x_1, x_2\} > 0$ and $x_1 \neq x_2$, bargaining power is strictly increasing in time for one player, and strictly decreasing for the other.*

Proof. The bargaining power of $i \in \{1, 2\}$, at period $t + 1$, given that the conflict continues for t successive periods, is as follows due to Lemma 1:

$$B_i(x_1^{t+1}, x_2^{t+1}) = \frac{(x_i)^{(1+\alpha)^{t+1}\beta}}{(x_1)^{(1+\alpha)^{t+1}\beta} + (x_2)^{(1+\alpha)^{t+1}\beta}}$$

Then,

$$\begin{aligned}
B_i(x_1^{t+1}, x_2^{t+1}) &> B_i(x_1^t, x_2^t) \\
&\iff \\
\frac{(x_i)^{(1+\alpha)^{t+1}\beta}}{(x_1)^{(1+\alpha)^{t+1}\beta} + (x_2)^{(1+\alpha)^{t+1}\beta}} &> \frac{(x_i)^{(1+\alpha)^t\beta}}{(x_1)^{(1+\alpha)^t\beta} + (x_2)^{(1+\alpha)^t\beta}}
\end{aligned}$$

Rearranging and simplifying the expression above, I get:

$$B_i(x_1^{t+1}, x_2^{t+1}) > B_i(x_1^t, x_2^t) \iff x_i > x_j$$

The result also implies,

$$B_i(x_1^{t+1}, x_2^{t+1}) > B_i(x_1^t, x_2^t) \iff B_j(x_1^{t+1}, x_2^{t+1}) < B_j(x_1^t, x_2^t)$$

\square

Lemma 2 suggests that there are decreasing (increasing) returns to fighting for the player with fewer (larger) initial forces. Given the result, it is easy to deduce that each player has orthogonal incentives for continuing a conflict. Next I am going to show that for the player with the fewer forces, opting for conflict is a weakly dominated strategy.

Lemma 3.5. *Given, $\max \{x_1, x_2\} > 0$ and $x_1 \neq x_2$, then any $s_j : a_j^t = F$, for any $t \in \{0, 1, \dots\}$, is a weakly dominated strategy for $j \in \{1, 2\}$ such that $x_j < x_i$ and $i \neq j$.*

Proof. Assume $x_1 > x_2$ and $a_1^t = F$. Then, $a_2^t = P$ and $a_2^t = F$ are payoff equivalent for player j , while both lead to fighting for one more stage.

Now assume $a_1^t = P$. If $a_2^t = P$, conflict ends and the current payoff player two obtains is $B_2(x_1^t, x_2^t) v_2$.

On the other hand, if conflict continues, both players get zero payoff at t and by Lemma 2, player two obtains the bargaining power $B_2(x_1^{t+1}, x_2^{t+1})$ in the next stage. As v_2 is constant, I may ignore it in the maximization problem:

$$V(B_2(x_1^t, x_2^t)) = \max_{\{P, F\}} \{B_2(x_1^t, x_2^t), \delta V(B_2(x_1^{t+1}, x_2^{t+1}))\}$$

Let us simplify the notation as $B_2^t = B_2(x_i^t, x_j^t)$. The solution to this problem will be of the form,

$$V_2(B_2^t) = \begin{cases} B_2^t & \text{if } \bar{B} \leq B_2^t \\ \bar{B} = \delta V(B_2^{t+1}) & \text{if } \bar{B} > B_2^t \end{cases}$$

Assume now $\bar{B} \geq B_2^t$. Then,

$$\bar{B} = \delta B_2^{t+1} \Rightarrow B_2^{t+1} \geq \bar{B}$$

Which holds as $\delta \in (0, 1)$. However by Lemma 2, $B_2^t > B_2^{t+1}$, which contradicts with $\bar{B} \geq B_2^t$. Thus, there is no $\bar{B} > B_2^t$ satisfying $V_2(B_2^t)$. Therefore

$$P = \operatorname{argmax}_{\{P, F\}} \{B_2(x_1^t, x_2^t), \delta V(B_2(x_1^{t+1}, x_2^{t+1}))\}$$

□

Lemma 3.6. *Given $\max \{x_1, x_2\} > 0$, $x_1 \neq x_2$, and no player uses a weakly dominated strategy, there is no SPNE in which the conflict lasts forever.*

Proof. Assume $x_1 > x_2$. By Lemma 3, $s_2 = \{P, P, \dots, P\}$ is a weakly dominant strategy for player two.

Now assume that player one ends the game at $t \geq 0$ by $a_1^t = P$ -as unanimous selection of P ends the game- and obtains the discounted payoff,

$$\delta^t B_1(x_1^t, x_2^t) v_1 - x_i$$

The value of this discounted payoff as $t \rightarrow \infty$ is:

$$\lim_{t \rightarrow \infty} \delta^t B_1(x_1^t, x_2^t) v_1 \leq \lim_{t \rightarrow \infty} \delta^t v_1 = 0$$

as $B_1(x_1^t, x_2^t) \in (0, 1)$, $\forall t$, and $v_1 > 0$ and $\delta \in (0, 1)$.

Whereas by terminating the conflict at some $t < \infty$ player one obtains the payoff

$$\delta^t B_1(x_1^t, x_2^t) v_1 > 0 = \lim_{t \rightarrow \infty} \delta^t B_1(x_1^t, x_2^t) v_1$$

Therefore terminating the conflict at a $t < \infty$ is more profitable for player one. \square

Using elimination of weakly dominated strategies and assuming that $x_1 > x_2$; Lemmas 2-4 imply that the equilibrium timing of the termination of the conflict is given by the solution to the following optimization problem of the player with the larger initial forces, i.e. player one.

$$\max_{t \in \{0, 1, 2, \dots\}} g_1(x_1, x_2, t) v_1 = \delta^t B_1^t(x_1, x_2) v_1$$

such that

$$B_1^t(x_1, x_2) = \frac{(x_1)^{(1+\alpha)^t \beta}}{(x_1)^{(1+\alpha)^t \beta} + (x_2)^{(1+\alpha)^t \beta}} \quad (3.9)$$

$$\max\{x_1, x_2\} > 0, x_1 \neq x_2$$

Notice that if $\hat{t} = \arg \max_t g_i(x_1, x_2, t)$ then $s^{\hat{t}} = (P, P)$ and $s^{\hat{t}-t} \neq (P, P)$ for $\hat{t} - t \geq 0$.

Next I are going to characterize the solution to problem (3.9).

Lemma 3.7. *There is a solution $\hat{t} \in \{0, 1, 2, \dots\}$ for the problem (3.9).*

Proof. As x_1 and x_2 are given in the subgame that starts after the initial stage, I denote $g_1(x_1, x_2, t) = g_1(t)$ and,

$$g_1(t) = \delta^t \frac{(x_1)^{(1+\alpha)^t \beta}}{(x_1)^{(1+\alpha)^t \beta} + (x_2)^{(1+\alpha)^t \beta}} v_1 = \delta^t \frac{1}{1 + r^{(1+\alpha)^t \beta}}$$

Where $r = \frac{x_2}{x_1}$, and $r \in [0, 1)$ as $x_1 > x_2$. As, $\delta \in (0, 1)$:

$$\lim_{t \rightarrow 0} g_1(t) = \frac{1}{1 + r^\beta}$$

$$\lim_{t \rightarrow \infty} g_1(t) = 0$$

Moreover $g(t)$ is bounded above by 1, bounded below by 0, and continuously differentiable. Then, by Weierstrass theorem, there is a $\hat{t} \in [0, t_\varepsilon] \cap \{0, 1, 2, \dots\}$ such that $\hat{t} \in \operatorname{argmax}_t g_i(t)$. \square

To simplify, I assume that the domain for t is continuous and $t \in [0, \infty)$ for the time being. Note that the maximizer to this problem for positive integers would be either one of the neighboring integers of the maximizer in real positive numbers or zero. This is because, by the continuity of the objective function, in a closed neighborhood of the maximizer the objective function is increasing for every point smaller than the maximizer and decreasing for every point larger than the maximizer.

Theorem 3.8. *The solution*

$$t^* = \operatorname{argmax}_{\{t \geq 0\}} \delta^t \frac{1}{1 + r^{(1+\alpha)^t \beta}}$$

is given by $t^* \in T(k) = \left\{ \frac{\log(\Psi(k)) - \log(\beta)}{\log(1+\alpha)}, k \in \{-1, 0\} \right\}$ and

$$\Psi(k) = \frac{\log(1+\alpha) W_k \left(\frac{\delta^{-\frac{1}{\log(1+\alpha)}} \log(\delta)}{\log(1+\alpha)} \right) + \log(\delta)}{\log(1+\alpha) \log(r)} \quad (3.10)$$

where $W_k(x)$ is the Lambert-W function.

- i If $T(k) = \emptyset$ there is no maximizer $t^* > 0$, thus $t^* = 0$
- ii If $\|T(k)\| = 1$, $t^* = T(-1)$
- iii If $\|T(k)\| = 2$, t^* is the larger member, $T(-1)$, or $t^* = 0$

Proof. The solution to the problem requires first the fulfillment of the following condition obtained by the first order conditions:

$$F(\psi) = \log(\delta) (1 + r^\psi) - \psi \log(1+\alpha) \log(r) r^\psi = 0$$

where $(1+\alpha)^t \beta = \psi$. Note that $\psi > \beta > 0$. This problem has at most two real solutions, for $k \in \{-1, 0\}$, given by (3.10) as the Lambert-W function only attains real values for $k \in \{-1, 0\}$.

- i For $x < -\frac{1}{e}$, $W_{-1}(x)$ and $W_0(x)$ are not defined so $F(\psi) \neq 0$ for any ψ . Moreover $\lim_{\psi \rightarrow \infty} F(\psi) = \log(\delta) < 0$. As F is continuous it then never becomes positive, therefore the objective function is strictly decreasing.

- ii If $F(1) < 0$ and as $\lim_{\psi \rightarrow \infty} F(\psi) < 0$ it needs at least two points of intersection with the x -axis, leading to a contradiction. Then $F(1) > 0$ and $\exists t^{**}$ such that $F(t) > 0$ for all $t < t^{**}$ and $F(t) < 0$ for all $t > t^{**}$. Thus, t^{**} has to be the maximizer, i.e. $t^* = t^{**}$.
- iii If $F(1) > 0$ and as $\lim_{\psi \rightarrow \infty} F(\psi) < 0$ it needs an odd number of points of intersection with the x -axis, leading to a contradiction. Then $F(1) < 0$ and $\exists t', t''$ with $t' < t''$ such that $F(t) < 0$ for all $t < t'$, $F(t) > 0$ for all $t \in (t', t'')$, and $F(t) < 0$ for all $t > t''$ and $t'' = T(-1)$ as $|W_{-1}(x)| > |W_0(x)|$ for all $x \in (-\frac{1}{e}, 0)$. But as F decreases first, $t'' = t^* \iff g(t'') \geq g(0)$.

□

Note that the proof of Theorem 1 implies the sufficiency of checking the neighboring integers of the maximum and point zero as function F can only have two real roots: one (global) maximum and one (global) minimum.

Proposition 3.9. *There is a threshold discount rate $0 < \hat{\delta} < 1$ such that, for all $\delta < \hat{\delta}$ war would not begin and this threshold decreases with the decisiveness parameter α .*

Proof. For war not to begin for sure the Lambert-W function should not have any real roots. Therefore

$$\frac{\delta^{-\frac{1}{\log(1+\alpha)}} \log(\delta)}{\log(1+\alpha)} < -1/e$$

The left hand side is always negative and implies the following condition for the discount factor:

$$\delta < (1+\alpha) W_0 \left[\left(\frac{1}{(1+\alpha)} \right)^{\frac{1}{\log(1+\alpha)}} \right]^{\log(1+\alpha)} = \hat{\delta} \quad (3.11)$$

The first derivative of the right hand side, $[\log(W_0(\frac{1}{e})) + 1] (\log(W(\frac{1}{e})))^{\log(1+\alpha)}$ is negative by the definition of the Lambert-W function. □

Proposition 3.10. *Assume $\delta > \hat{\delta}$. Then, $\exists R = [r', r''] \subset [0, 1]$ with $1 > r'' > r' > 0$ such that war takes place if and only if $r \in R$.*

Proof. By Theorem 1, $t^* = 0$ if $\psi < \beta$. Note that by equation (3.10), $\frac{d\psi}{dr} = \frac{K}{r \log^2(r)} > 0$ where $K > 0$ and $\lim_{r \rightarrow 0} g(\psi(r)) = 0 \leq \beta$. As $\psi(r)$ is continuously differentiable, $\exists r' \in [0, 1]$ such that, for all $r < r'$, $t^* = 0$.

Now take the maximized value function $g(\psi(r)) = \delta^{t(r)} \frac{1}{1+r^{\psi(r)}}$.

$$\frac{\partial g(\psi(r))}{\partial r} = \delta^{t(r)} \left\{ \log(\delta) t'(r) \frac{1}{1+r^{\psi(r)}} - [\log(r) \psi'(r) + \psi(r)] \right\} < 0$$

as $t'(r) > 0$, which is implied by $\psi'(r) > 0$, and the second term of curly brackets is zero by $\frac{d\psi}{dr} = \frac{K}{r \log^2(r)}$ and $\psi(r) = -\frac{K}{\log(r)}$. Moreover, $\lim_{\psi \rightarrow \infty} g(\psi) = 0$ and $g(t=0) = \frac{1}{1+r^\beta}$. As $\psi(r)$ is continuously differentiable, $\exists r'' \in [0, 1]$ such that, for all $r > r''$, $t^* = 0$. \square

Theorem 1 completely characterizes the set of parameters that leads to war. First to be deduced is that the discount factor has to be sufficiently large for war to begin at all, which is intuitive as the future gains of damage will be of negligible value. Moreover, if the war technology is not very effective in destruction, which implies a small α , the discount factor has to be even larger to achieve some advantage in the future periods.

Secondly, by Proposition 3, I see that war will not payoff even for a large discount factor if the relative advantage in forces for one player exists, but either too large or too small. Since destruction of a small force will not push the benefits to a sufficiently larger bargaining power, starting the war will not pay off. On the other hand, if the advantage in forces is too small, to achieve a better bargaining position will last too long and will be discounted heavily. Therefore, war will not begin at all either. I may see this relation in Figure 1.

Third, by Propositions 1 and 4, war lasts longer if the relative advantage of a player is small. This result is implied simply by $t'(r) > 0$, i.e. the optimal time to stop increases as r increases which implies symmetric forces. However, if this advantage is smaller than a threshold value, given by Proposition 3, war will not start at all. This result follows simply because the destruction of a stronger rival force requires a longer path of battles.

Finally, proposition 1 implies that, optimal timing to stop decreases with the effectiveness of the forces in bargaining power, β . I can observe that by $\frac{\partial t(\cdot)}{\partial \beta} = -\frac{1}{\beta}$. The intuition of this result is, the more the forces are effective in gaining bargaining power the more the incentives to stop the conflict as soon as possible.

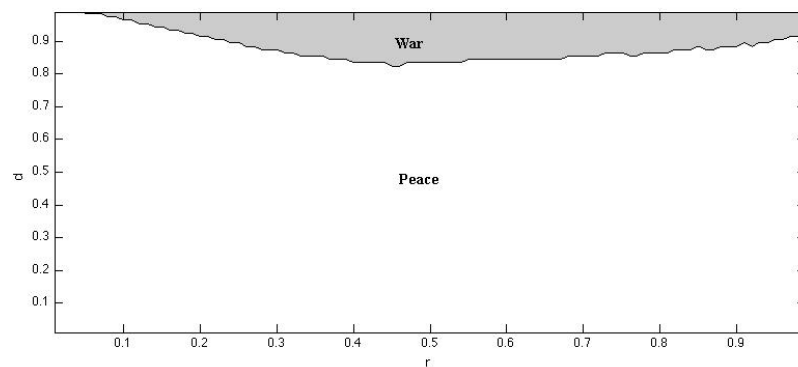


Figure 3.1: War and peace as a function of δ and r ($\alpha = \beta = 1$)

3.4 Subgame perfect Nash equilibrium

In this section I analyze first stage choices of forces. Due to Proposition 1, I already know that the armed conflict ends at a period t^* , such that $t^* \in \{0, 1, 2, \dots\}$. Then, the problem of each player at the initial stage is maximizing his payoff:

$$\max_{\{a_i=x_i \geq 0\}} v_i \delta^{t^*} \frac{x_i^{\psi(t^*)}}{x_i^{\psi(t^*)} + x_j^{\psi(t^*)}} - x_i$$

As t^* is the outcome of the equilibrium of the subgame, the optimal action, x_i , will be a part of the Subgame Perfect equilibrium of the whole game.

Proposition 3.11. *If β is such that $v_1^\beta + v_2^\beta > \beta v_1^\beta$ and $\delta \leq \hat{\delta}$, the Subgame Perfect equilibrium is $s^* = (s_1^*, s_2^*)$ such that $x_i^* = \frac{\beta v_i^{\beta+1} v_j^\beta}{(v_1^\beta + v_2^\beta)^2}$ and $a^t = P$, for all t .*

Proof. By Theorem 1, war does not begin. Thus, for any t , the optimum action is $a_i^t = P$. Therefore, using the definition of $\psi = (1 + \alpha)^t \beta$, the optimum action in the initial stage is given by:

$$\operatorname{argmax}_{x_i \geq 0} v_i \frac{x_i^\beta}{x_i^\beta + x_j^\beta} - x_i$$

which implies the equilibrium values given. \square

The condition on β is the requirement for the existence of a pure strategy Nash equilibrium found in [Nti \(1999\)](#). The equilibrium values also follow from [Nti \(1999\)](#).

Next I am going to assume that $\delta \geq \hat{\delta}$. Thus, there is an interval of relative arming, $R \subset (0, 1)$, such that war takes place. The difference of this case from the previous equilibrium described is that a change in a player's action in the initial stage also effects the stopping time of war in the subgame.

Lemma 3.12. *Assume $x_i^* = \frac{\psi v_i^{\psi+1} v_j^\psi}{(v_1^\psi + v_2^\psi)^2}$, then there is no profitable deviation to a strategy $x_i \neq x_i^*$ such that $\psi(x_i, x_j^*) = \psi(x_i^*, x_j^*)$.*

Proof. As time is discrete, there is a continuum of values $R_t = [r_{t-1}, r_{t+1}]$ such that for all $r \in R_t$ optimal time to stop is t . Therefore, ψ is constant for all $r \in R_t$. As x_i^* is the equilibrium value for that ψ , there is no profitable deviation in a closed neighborhood of x_i^* . \square

The lemma above reduces the set of candidate equilibria to equilibrium strategy profiles for different decisiveness parameters, ψ , which is a function of optimal time to stop. I now define a

best response strategy for a player.

$$x_i^{**}(x_j) = \operatorname{argmax}_{\{x_{i0}, x_{i1}, \dots, x_{iT}\}} \left\{ v_i \frac{x_{i0}^{\psi(r_0)}}{x_{i0}^{\psi(r_0)} + x_j^{\psi(r_0)}} - x_{i0}, \right. \\ \left. v_i \delta \frac{x_{i1}^{\psi(r_1)}}{x_{i1}^{\psi(r_1)} + x_j^{\psi(r_1)}} - x_{i1}, \dots, v_i \delta^T \frac{x_{iT}^{\psi(r_T)}}{x_{i0}^{\psi(r_T)} + x_j^{\psi(r_T)}} - x_{iT} \right\} \quad (3.12)$$

where $x_{it}(x_j) = \operatorname{argmax}_{x_i \geq 0} v_i \delta^t \frac{x_i^{\psi(r_t)}}{x_i^{\psi(r_t)} + x_j^{\psi(r_t)}} - x_i$

Proposition 3.13. *Given $\delta > \hat{\delta}$, $(x_1^*, x_2^*) = \left(\frac{\beta v_1^{1+\beta} v_2^\beta}{(v_1^\beta + v_2^\beta)^2}, \frac{\beta v_1^\beta v_2^{1+\beta}}{(v_1^\beta + v_2^\beta)^2} \right)$ is an equilibrium strategy profile if and only if*

$$\frac{\log(\delta)}{\log(1+\alpha) [\log(v_2) - \log(v_1)]} \leq 1 + \left(\frac{v_2}{v_1} \right)^\psi \quad (3.13)$$

where $\psi = (1+\alpha)\beta$.

Proof. Using the previous denotation $r = \frac{x_2^*}{x_1^*}$, I may express the payoff function of player two as:

$$u_2(r) = \delta^{t(r(x_2))} \frac{r(x_2)^{\psi(r(x_2))}}{1 + r(x_2)^{\psi(r(x_2))}} - x_2$$

First, by Lemma 6 there is no profitable deviation in the neighborhood of x_2^* for player two. Moreover, at the candidate equilibrium $t(r) = 0$. Thus, reducing x_2 will not payoff. Now:

$$\frac{\partial u_2}{\partial x_2} = \delta^t \left\{ \frac{\log(\delta) r'(x_2) t'(r)}{1 + r^\psi} - \frac{r^{\psi-1} r'(x_2) [\psi + \log(r) r \psi'(r)]}{(1 + r^\psi)^2} \right\} - 1$$

First term in the curly brackets is negative as $r'(x_2) > 0$ and $\delta \in (0, 1)$. Second term is zero by definitions of $\psi(r)$ and $\psi'(r)$. Therefore an increase that in the action x_2 , which leads to an increase in the decisiveness parameter, makes player two worse off.

On the other hand the payoff function of player one is:

$$u_1(r) = \delta^{t(r(x_1))} \frac{1}{1 + r(x_1)^{\psi(r(x_1))}} - x_1$$

Note that player one has no incentive to increase x_1 as that would lead a further reduction in r and would not affect t in the subgame. Moreover,

$$\frac{\partial u_1}{\partial x_1} = \delta^t \left\{ \frac{\log(\delta) r'(x_1) t'(r)}{1 + r^\psi} - \frac{r^{\psi-1} r'(x_1) [\psi + \log(r) r \psi'(r)]}{(1 + r^\psi)^2} \right\} - 1$$

First term in the curly brackets is positive as $r'(x_1) < 0$. Then, decreasing x_1 is not optimal for player one if the first term is larger than 1, i.e. marginal decrease in the cost is smaller than the marginal decrease in benefit of waiting one more period or:

$$1 > \frac{\log(\delta) r'(x_1) t'(r)}{1 + r^\psi}$$

Using definitions of r , $r'(x)$, $t'(r)$ (by Proposition 1), and in the equilibrium $r = \frac{v_2}{v_1}$ imply equation (13). \square

Proposition 5 implies that the equilibrium outcome of this game of dynamic conflict is armed peace, given that equation (13) is satisfied. Equation (13) enforces that, to sustain a peaceful equilibrium, discount factor should be sufficiently small and the effectiveness of armed forces in destruction should be sufficiently small. These are intuitive as otherwise, player one would be willing to engage in war as he could destroy most of the forces possessed by player two and he would value the bargaining power achieved after this one stage of war greatly. Equation (13) also implies that the valuations of the players cannot be very similar.

It is extremely complicated to show that there is no subgame perfect equilibrium with war in the stage after the initial stage ends. However, in order demonstrate that it is hard to sustain such a subgame perfect equilibrium I will consider the following example.

Example 3.1. I take $\delta = 0.91$, $v_1 = 10$, $v_2 = 1.8$ and $\alpha = \beta = 0.5$. Given these parameters, the length of war as a function of relative forces $r \in (0, 1)$ is shown in Figure 2. Now assume

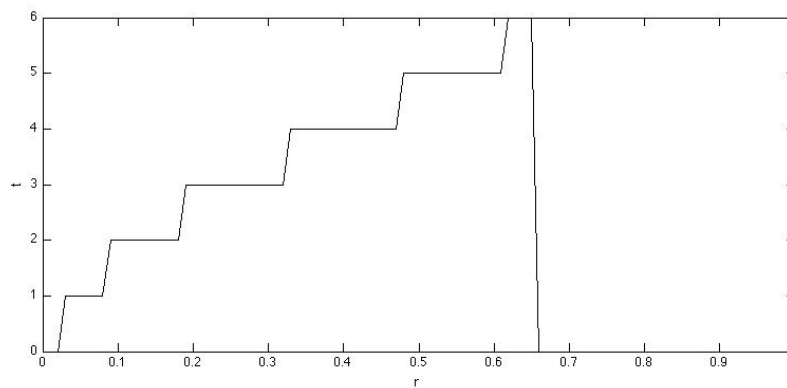


Figure 3.2: Length of war as a function of r

that there is an equilibrium with $r^* = 0.18$ so that war lasts for one period and $\psi = 0.75$. In this equilibrium, using Lemma 6, the equilibrium force is $\frac{\psi v_i^{1+\psi} v_j^\psi}{(v_1^\psi + v_2^\psi)^2}$ and equilibrium utility is

$\frac{\psi v_i^{1+\psi}}{(v_1^\psi + v_2^\psi)^2} \left(v_i^\psi + (1 - \psi) v_j^\psi \right)$. Equilibrium implies also that $\frac{v_2}{v_1} = 0.18$.

These imply $x_1^ = 0.472$, $x_2^* = 0.085$, $u_1^* = 6.658$, and $u_2^* = 0.270$. Now in order to reduce the length of the conflict to $t = 0$ relative forces has to decline to $r' = 0.08$. For player two to achieve this, he has to reduce his forces at least to $x_2' = 0.038$ which decreases ψ to 0.5 hence increasing his payoff to $u_2' = 0.35$ given x_1^* . Thus a subgame perfect equilibrium with initial stage actions x_1^* and x_2^* cannot be supported with war lasting one stage.*

3.5 Conclusions

In this paper I considered a dynamic model of war that include initial arming decisions in the shadow of a dynamic conflict in which adversaries decide whether to stop fighting or not at each stage depending on the current armed forces they possess.

I found, given the initial stage of arming, that war will not begin if players heavily discount future payoffs or the relative advantage of forces is either very small or very large for one adversary. Moreover, war is elongated if the advantage is small but less than a threshold value. These results imply that very similar or very asymmetric forces do not tend to have a conflict as the gains in these situations are not worth starting a war. On the other hand, in case war starts at all, similar forces tend to have longer wars than asymmetric ones.

I also find that in the pure strategy subgame perfect equilibrium of the game that the outcome of this game is armed peace. This outcome emanates from the ability of the player with lower valuation to invest less in forces so that he can avoid a depreciation in his bargaining power.

In the cases where arming decisions are made in a prospect of war, the equilibrium predicts armed peace. However, countries usually invest in arming as defensive measures. Therefore, the theoretical predictions of the length of a war can be valid for the cases in which belligerents already have a stock of armed forces given.

Appendix A

Chapter 1

A.1 Existence of SPNE with Generalized Tullock CSF

Assume the CSF is given as follows:

$$p_1 = \frac{g_1^s}{g_1^s + \theta g_2^s} \quad \text{and} \quad p_2 = \frac{\theta g_2^s}{g_1^s + \theta g_2^s}. \quad (\text{A.1})$$

Then, the payoff functions of war for each player are:

$$u_1 = V_1 \frac{g_1^s}{g_1^s + \theta g_2^s} - g_1 \quad \text{and} \quad u_2 = V_2 \frac{\theta g_2^s}{g_1^s + \theta g_2^s} - g_2. \quad (\text{A.2})$$

The FOCs of the payoff maximization are:

$$\frac{\partial u_1}{\partial g_1} = \frac{\partial u_2}{\partial g_2} = V_1 \frac{s\theta g_1^{s-1} g_2^s}{(g_1^s + \theta g_2^s)^2} - 1 = V_2 \frac{s\theta g_2^{s-1} g_1^s}{(g_1^s + \theta g_2^s)^2} - 1 = 0 \quad (\text{A.3})$$

Solving system (A.3), we obtain:

$$g_1^* = V_1 \frac{s\theta \gamma^s}{(1 + \theta \gamma^s)^2} \quad \text{and} \quad g_2^* = V_2 \frac{s\theta \gamma^s}{(1 + \theta \gamma^s)^2} \quad (\text{A.4})$$

These efforts are not necessarily an NE, because the Second Order Condition (SOC) of payoff maximization might not hold, so an extra argument is necessary. Consider player one. Her payoff function with $g_2 = g_2^*$ is continuous, so there is a maximum of this function over the interval $[0, V_1]$. Even though the maximization on the definition of an NE is on the real line, in equilibrium, no rational player will spend more on war than her valuation. Thus, the aforementioned maximum can be either located at the extremes, i.e. $g_1^* = 0$, $g_1^* = V_1$ or in interior, in which case this maximum is g_1^* . Payoffs for the first two options are zero and negative, respectively, so if we show that payoffs for player one are non-negative when evaluated at g_1^* and g_2^* , then g_1^* is

a best reply to g_2^* . The same argument applies for player two. Letting $V_2 = \gamma V_1$, we need:

$$V_1 \frac{1 + \theta\gamma^s - s\theta\gamma^s}{(1 + \theta\gamma^s)^2} \geq 0 \text{ and } V_2 \frac{\theta\gamma^s + \theta^2(\gamma^s)^2 - s\theta\gamma^s}{(1 + \theta\gamma^s)^2} \geq 0 \quad (\text{A.5})$$

or

$$1 + \theta\gamma^s \geq s\theta\gamma^s \quad \text{and} \quad \theta + \theta^2\gamma^s \geq s\theta \quad (\text{A.6})$$

which when $\theta = 1$ boil down to the conditions in [Nti \(1999\)](#). Note that when $s \leq 1$, (A.6) always holds. Finally, when both players are identical ($\theta = \gamma = 1$), (A.6) reads that $s \leq 2$.

Some remarks are in order. First, in the NE constructed above,

$$\frac{g_2^*}{g_1^*} = \frac{V_2}{V_1} = \gamma \quad (\text{A.7})$$

Thus, as it happens when $s \leq 1$, war productivity does not affect the ratio of equilibrium expenses.

Second, if (A.6) does not hold for at least one player, the second stage of UR will fail to have an NE in pure strategies. Even though there is an equilibrium in mixed strategies, given the difficulty of interpreting mixed strategies in our framework, we do not pursue this matter.

Finally, using (A.6), we can show that the conditions for peace to hold in equilibrium are

$$\frac{1 + (1 - s)\theta\gamma^s}{(1 + \theta\gamma^s)^2} V_1 \leq \varepsilon V_1 \quad \text{and} \quad \frac{\theta(\gamma^s)^2 + (1 - s)\theta\gamma^s}{(1 + \theta\gamma^s)^2} V_2 \leq (1 - \varepsilon) V_2$$

The conditions above boil down to the following set of divisions admitting peace.

$$\frac{1 + (1 - s)\theta\gamma^s}{(1 + \theta\gamma^s)^2} \leq \varepsilon \leq \frac{1 + (1 + s)\theta\gamma^s}{(1 + \theta\gamma^s)^2}$$

which for $s = 1$ are identical to (1.13).

A.2 Resource Constraints under Complete Information

Assume players one and two, respectively, own resources $R_1 = R$ and $R_2 = aR$ to be used in war, exclusively, where $a \geq 0$ is the relative amount of resources of player two. Assume $R \geq g_1$, $aR \geq g_2$, so no player can spend more than she initially owns. We now analyze both UR and FDR under this new assumption.

If player i is constrained, the marginal utility of effort is greater than the marginal cost of it at $g_i = R_i$. Formally

$$\left. \frac{\partial u_i}{\partial g_i} \right|_{g_i=R_i} = V_i \frac{\theta g_j}{(g_1 + \theta g_2)^2} - 1 \geq 0 \quad (\text{A.8})$$

If (A.8) holds, the equilibrium effort of player i is $g_i^* = R_i$. Given the concavity of the payoff function, this condition is necessary and sufficient.

First, consider UR. There are four cases. In the first one, no player is resource-constrained, which is the case we analyzed in the main text. In the second one, both players are resource-constrained, which implies equilibrium efforts are $g_1^* = R$, and $g_2^* = aR$. Using (A.8), this occurs if and only if

$$V_1/R \geq (1 + a\theta)^2 / a\theta \quad \text{and} \quad V_2/R \geq (1 + a\theta)^2 / \theta \quad (\text{A.9})$$

Note that (A.9) implies that, in an NE, payoffs are strictly positive. In UR, the payoff of peace is zero; hence, war is the only equilibrium outcome when both countries are constrained. Clearly, this result extends to the cases where only one player is constrained.

Now consider FDR. If peace holds in equilibrium, both countries are better off choosing peace:

$$V_1 \frac{1}{1 + a\theta} - R \leq \varepsilon V_1 \quad \text{and} \quad V_2 \frac{a\theta}{1 + a\theta} - aR \leq (1 - \varepsilon) V_2 \quad (\text{A.10})$$

or

$$\frac{1}{1 + a\theta} - \frac{R}{V_1} \leq \varepsilon \leq 1 + \frac{aR}{V_2} - \frac{a\theta}{1 + a\theta} \quad (\text{A.11})$$

For peace to hold, the LHS of (A.11) has to be less than 1 and the RHS of (A.11) has to be greater than 0, as $\varepsilon \in (0, 1)$. These conditions are implied by (A.9). Clearly, the RHS of (A.11) has to be larger than the LHS of it. Suppose on the contrary that

$$1 + \frac{aR}{V_2} - \frac{a\theta}{1 + a\theta} \leq \frac{1}{1 + a\theta} - \frac{R}{V_1} \Leftrightarrow \frac{aR}{V_2} + \frac{R}{V_1} \leq 0 \quad (\text{A.12})$$

which is impossible. Thus, there is an ε such that (A.11) holds and peace is the only equilibrium outcome. Now assume that only one player, say player one, is constrained. Thus, $g_1^* = R_1$. The best reply of player two is

$$g_2^* = \frac{\sqrt{\theta V_2 g_1} - g_1}{\theta} = \frac{\sqrt{\theta R V_2} - R}{\theta} \quad (\text{A.13})$$

For peace to hold, the distribution of the resource $\varepsilon \in (0, 1)$ should satisfy:

$$V_1 \sqrt{\frac{R}{\theta V_2}} - R \leq \varepsilon V_1 \quad \text{and} \quad V_2 - 2\sqrt{\frac{R V_2}{\theta}} + \frac{R}{\theta} \leq (1 - \varepsilon) V_2 \quad (\text{A.14})$$

Using simple algebra, we reduce (A.14) to the following.

$$\sqrt{\frac{R}{\theta V_2}} - \frac{R}{V_1} \leq \varepsilon \leq 2\sqrt{\frac{R}{\theta V_2}} - \frac{R}{\theta V_2} \quad (\text{A.15})$$

The LHS (A.15) has to be less than 1. Assume otherwise. Note at g_1^* and g_2^* (A.8) reads:

$$\frac{V_1}{\sqrt{\theta R V_2}} \left(1 - \sqrt{\frac{R}{\theta V_2}} \right) \geq 1$$

However, the LHS being larger than 1 contradicts the condition above. The necessity of the RHS to be larger than 0 is implied again by condition (A.8).

The last condition is the non-emptiness of the set defined by (A.15). Suppose this interval is empty. Then:

$$\sqrt{\frac{R}{\theta V_2}} - \frac{R}{\theta V_2} + \frac{R}{V_1} \leq 0 \Leftrightarrow \sqrt{\frac{R V_2}{\theta}} - \frac{R}{\theta} \leq -\frac{R V_2}{V_1} \quad (\text{A.16})$$

whereas by the FOC of payoff maximization of player one, we obtain that

$$\sqrt{\frac{R V_2}{\theta}} - \frac{R}{\theta} \geq \frac{R V_2}{V_1} \quad (\text{A.17})$$

which together with (A.16) implies a contradiction. Thus, there is always a division of the resource such that peace is the unique equilibrium outcome. The case when country 2 is constrained and country 1 is not is equivalent. Summing up, proposition 2 can be generalized to include resource constraints.

A.3 Equilibrium in FDR with asymmetric information

Neglecting the non-negativity constraints, the FOCs of (1.17) for a high and a low type are, respectively:

$$\frac{\partial u_1}{\partial g_H} = V_H \frac{\theta g_2}{(g_H + \theta g_2)^2} - 1 = \frac{\partial u_1}{\partial g_L} = \rho V_H \frac{\theta g_2}{(g_L + \theta g_2)^2} - 1 = 0$$

Using the expressions above, we get the following best response functions for the high and the low type, respectively, as:

$$g_H = \sqrt{V_H \theta g_2} - \theta g_2 \quad \text{and} \quad g_L = \sqrt{\rho V_H \theta g_2} - \theta g_2$$

The FOC of (1.18) for country two is:

$$\frac{\partial u_2}{\partial g_2} = V_2 \theta \left[\mu \frac{g_H}{(g_H + \theta g_2)^2} + (1 - \mu) \frac{g_L}{(g_L + \theta g_2)^2} \right] - 1 = 0$$

Equilibrium war efforts (1.19)-(1.21) are found by substituting the best response functions of the two possible types into the FOC of the problem of country two:

$$\frac{\partial u_2}{\partial g_2} = V_2 \theta \left[\mu \frac{\sqrt{V_H \theta g_2} - \theta g_2}{V_H \theta g_2} + (1 - \mu) \frac{\sqrt{\rho V_H \theta g_2} - \theta g_2}{\rho V_H \theta g_2} \right] - 1 = 0$$

Using the notation $x = \theta \gamma$, we solve the equation above for g_2 , and we find the equilibrium war effort of country two as:

$$g_2^* = \frac{\rho V_H}{\theta} \left[\frac{1 - \mu (1 - \sqrt{\rho})}{\frac{\rho}{x} + 1 - \mu (1 - \rho)} \right]^2$$

Substituting the result above into the best response functions of the high and the low type, respectively:

$$g_H^* = \sqrt{\rho} V_H \left[\frac{1 - \mu (1 - \sqrt{\rho})}{\frac{\rho}{x} + 1 - \mu (1 - \rho)} \right] - \rho V_H \left[\frac{1 - \mu (1 - \sqrt{\rho})}{\frac{\rho}{x} + 1 - \mu (1 - \rho)} \right]^2$$

$$g_L^* = \rho V_H \frac{1 - \mu (1 - \sqrt{\rho})}{\frac{\rho}{x} + 1 - \mu (1 - \rho)} - \rho V_H \left[\frac{1 - \mu (1 - \sqrt{\rho})}{\frac{\rho}{x} + 1 - \mu (1 - \rho)} \right]^2$$

Simplifying the expressions above, we find the following equilibrium war efforts of the high and the low type:

$$g_H^* = \sqrt{\rho} V_H \frac{\left[\frac{\rho}{x} + (1 - \mu) (1 - \sqrt{\rho}) \right] [1 - \mu (1 - \sqrt{\rho})]}{\left[\frac{\rho}{x} + 1 - \mu (1 - \rho) \right]^2}$$

$$g_L^* = \rho V_H \frac{\left[\frac{\rho}{x} - \mu \sqrt{\rho} (1 - \sqrt{\rho}) \right] [1 - \mu (1 - \sqrt{\rho})]}{\left[\frac{\rho}{x} + 1 - \mu (1 - \rho) \right]^2}$$

A.4 Separating Equilibrium for FDR

First we show, whenever $\rho \geq .184$ or $x \geq .28$, condition (1.39) holds. Notice if $\rho = .184$ and $x = .28$, assumption (1.15) holds, and (1.39) is equivalent to:

$$G(\rho, x) = \left[1 - \frac{x}{\sqrt{\rho}(1+x)} \right]^2 - \frac{\rho(\rho+2x)}{(\rho+x)^2} \leq 0 \quad (\text{A.18})$$

We see that $G(\rho, x) \leq 0$ iff $H(\rho, x) \leq 0$, where the latter is defined as:

$$H(\rho, x) = (\rho + x) ((1+x) \sqrt{\rho} - x) - \rho(1+x) \sqrt{\rho+2x} \leq 0 \quad (\text{A.19})$$

Note that $H(\rho, 0) = 0$, so if $\frac{\partial H(\rho, x)}{\partial x} < 0$, (1.39) holds. Differentiating (A.19):

$$\sqrt{\rho} - (\rho + 2x) (1 - \sqrt{\rho}) - \frac{\rho(1+3x+\rho)}{\sqrt{\rho+2x}} \quad (\text{A.20})$$

The second and the third term in (A.20) are negative. Let us first disregard the second term. If $\frac{\partial H(\rho, x)}{\partial x} > 0$, we have

$$\rho^3 + 2\rho^2 + x(6\rho + 6\rho^2 - 2) + x^2(9\rho) < 0 \quad (\text{A.21})$$

When $x = 0$ or $x \rightarrow \infty$, (A.21) is impossible, so if (A.21) holds, there is an x for which the LHS of (A.21) is zero. Solving the equation:

$$x = \frac{2 - 6\rho - 6\rho^2 \pm 2\sqrt{3\rho^2 - 6\rho + 1}}{2(\rho^3 + 2\rho^2)} \quad (\text{A.22})$$

If $3\rho^2 - 6\rho + 1 < 0$ - which is true for $\rho \gtrsim .184$ - the expression under the root is negative so no solution for x exists.

Next, disregarding the third term in (A.20) and assuming $\frac{\partial H(\rho, x)}{\partial x} > 0$, we obtain:

$$\frac{\sqrt{\rho} - \rho + \rho\sqrt{\rho}}{2(1 - \sqrt{\rho})} > x \quad (\text{A.23})$$

If the inequality in (A.23) is reversed, which is possible as $\frac{\sqrt{\rho}}{1 - \sqrt{\rho}} > \frac{\sqrt{\rho} - \rho + \rho\sqrt{\rho}}{2(1 - \sqrt{\rho})}$, we arrive at a contradiction. Since the LHS of (A.23) is increasing in ρ and for $\rho \gtrsim .184$ we already proved that inequality (1.39) holds, it is sufficient that x is larger than the LHS of (A.23) evaluated at $\rho \gtrsim .184$ and this yields $x \gtrsim 0.28$.

Now let us partially differentiate $G(\rho, x)$ wrt ρ :

$$\begin{aligned} \frac{\partial G(., x)}{\partial \rho} &= \frac{2x^2}{(\rho + x)^3} - \frac{x}{(1 + x)\rho^{3/2}} + \frac{x^2}{\rho^2(1 + x)^2} \\ &\geq \frac{2x^2}{(1 + x)^3} - \frac{x}{(1 + x)\rho^{3/2}} + \frac{x^2}{(1 + x)^2} \\ &= \frac{x}{1 + x} \left(\frac{2x}{(1 + x)^2} - \frac{1}{\rho^{3/2}} + \frac{x}{1 + x} \right) \end{aligned}$$

So $\frac{2x}{(1+x)^2} - \frac{1}{\rho^{3/2}} + \frac{x}{1+x} > 0 \Rightarrow \frac{\partial G}{\partial \rho} > 0$. This inequality is equivalent to

$$\rho > \left(\frac{(1 + x)^2}{3x + x^2} \right)^{\frac{2}{3}} \quad (\text{A.24})$$

The RHS of (A.24) is less than one for $x > 1$, so if ρ is sufficiently close to one, (A.24) holds. Thus, when the uninformed country is powerful and there is very little uncertainty about the valuation of the informed country, an increase in ρ makes the interval larger, given π .

Now, we manipulate $\frac{\partial G(.,x)}{\partial \rho}$ to get:

$$\frac{\partial G(.,x)}{\partial \rho} = x \left(\frac{2x}{(\rho+x)^3} - \frac{1}{(1+x)\rho^{3/2}} + \frac{x}{\rho^2(1+x)^2} \right)$$

As $x \rightarrow 0$, the expression in brackets is negative, so $\frac{\partial G}{\partial \rho} < 0$. Therefore, when the uninformed country is very weak, an increase in ρ makes the interval smaller, given π .

Appendix B

Chapter 2

B.1 The sufficiency of local SOC

Proof. Assume $f(x_i) = x_i$. The SOC of the maximization problem of (2.5) is:

$$kx_2 \frac{-x_1^{k-2}(1-b)(2x_1+x_2) + kx_1^{k-2}(1-b) - bx_2^{k-1}(k+1)}{(x_1+x_2)^{k+2}}$$

The sign of this function depends on the numerator which reduces to:

$$x_1^{k-2}(1-b)((k-1)x_2 - 2x_1) - bx_2^{k-1}(k+1)$$

Given any $x_2 \geq 0$, and $k \leq 2$, the second term is constant and the first term is decreasing in x_1 . Therefore this function may either be positive at $x_1 \rightarrow 0$, but monotone decreasing to negative values as x_1 increases. As the second derivative changes sign only once. It may contain only one minimum and one maximum. Thus, the negativity of the local SOC are sufficient to guarantee a maximum.

The proof regarding $f(x_i) = x_i^\alpha$ is analogous as transforming $x_1^\alpha = \chi_1$ and $x_2^\alpha = \chi_2$, then the SOC conditions are the same except for the cost function which transform into a strictly convex function as $x_1 = \chi_1^{\frac{1}{\alpha}}$ and $x_2 = \chi_2^{\frac{1}{\alpha}}$. \square

B.2 The case of asymmetric players

Assume that the asymmetry stems from the constant marginal cost of effort, i.e. $c_i(x_i) = c_i x_i$. For convenience, we use $f(x_i) = x_i^\alpha$. Without loss of generality we assume $c_1 > c_2$. Then the

payoff function of player $i \neq j$ is:

$$u_i(x_i, x_j) = \left(\frac{x_i^\alpha}{x_i^\alpha + x_j^\alpha} \right)^k + b \left(1 - \frac{x_i^{\alpha k} + x_j^{\alpha k}}{(x_i^\alpha + x_j^\alpha)^k} \right) - c_i x_i \quad (\text{B.1})$$

Assuming that it exists, the equilibrium should satisfy the following relation that is given by the FOCs:

$$\frac{x_1^{\alpha-1}}{c_1} \left[(1-b) x_1^{\alpha(k-1)} x_2^\alpha + b x_2^{\alpha k} \right] = \frac{x_2^{\alpha-1}}{c_2} \left[(1-b) x_2^{\alpha(k-1)} x_1^\alpha + b x_1^{\alpha k} \right] = 1 \quad (\text{B.2})$$

Reducing the equation we find that

$$\frac{(1-b) x_1^{\alpha k-1} x_2^\alpha + b x_1^{\alpha-1} x_2^{\alpha k}}{(1-b) x_1^\alpha x_2^{\alpha k-1} + b x_1^{\alpha k} x_2^{\alpha-1}} = \frac{c_1}{c_2} \quad (\text{B.3})$$

Proposition B.1. *There exists no symmetric equilibrium for the game with asymmetric players.*

Proof. If $x_1 = x_2$ then the left hand side of the equation above becomes 1. As $c_1 > c_2$, a contradiction. \square

As $c_1 > c_2$, then by (B.3):

$$(1-b) \left\{ x_1^{\alpha k-1} x_2^\alpha - x_1^\alpha x_2^{\alpha k-1} \right\} > b \left\{ x_1^{\alpha k} x_2^{\alpha-1} - x_1^{\alpha-1} x_2^{\alpha k} \right\} \quad (\text{B.4})$$

For both sides of (B.4) to be positive, the following conditions have to hold:

$$x_1^{\alpha(k-1)-1} > x_2^{\alpha(k-1)-1} \quad (\text{B.5})$$

$$x_1^{\alpha(k-1)+1} > x_2^{\alpha(k-1)+1} \quad (\text{B.6})$$

Proposition B.2. *In equilibrium low cost player spends more in efforts if $\alpha(k-1) - 1 < 0$. Otherwise he spends more if $b \geq \frac{1}{2}$.*

Proof. First, take $\alpha(k-1) - 1 < 0$. Now let us assume $x_1 > x_2$. Then (B.5) is violated so LHS of (B.4) is negative. While, (B.6) holds so RHS is positive which contradicts (B.4). So if $\alpha(k-1) - 1 < 0$, which implies $k < 2$, in equilibrium low cost contestant spends more which holds as:

If $x_2 > x_1$, (B.5) holds so LHS of (B.4) is positive, and (B.6) is violated so the RHS of (B.4) is negative.

Now take $\alpha(k-1) - 1 > 0$. Assume $x_1 > x_2$, then both (B.5) and (B.6) hold. Therefore both sides are positive. Assume further that the term in curly bracket in the LHS of (B.4) is larger

than the term in curly bracket in the RHS of (B.4), which implies

$$x_1^{\alpha k-1} x_2^{\alpha-1} (x_2 - x_1) > x_1^{\alpha-1} x_2^{\alpha k-1} (x_1 - x_2)$$

and contradicts $x_1 > x_2$. So the term in curly bracket in the LHS of (B.4) is smaller than the term in curly bracket in the RHS of (B.4). Then (B.4) only holds if $b < \frac{1}{2}$ by the positivity of both sides of (B.4) \square

Note that we assumed existence of an equilibrium. Below we provide sufficient conditions for the solution to be an equilibrium.

First of all the SOC for this asymmetric case is equivalent to the symmetric case. Then at least for $k < 2$ the fulfillment of the local SOC is sufficient for a maximum.

We know that the sign of the SOC depends on the following function for players $i = 1, 2$:

$$x_1^{k-2} (1-b) ((k-1)x_2 - 2x_1) - bx_2^{k-1} (k+1) \quad (\text{B.7})$$

And symmetrically for player y :

$$x_2^{k-2} (1-b) ((k-1)x_1 - 2x_2) - bx_1^{k-1} (k+1) \quad (\text{B.8})$$

Recall that if $k < 2$ we guarantee that $x_2^* > x_1^*$ for (wlog) $c_1 > c_2$. Then it is clear that in equilibrium (B.8) is (locally) negative.

Now, a first sufficient condition for (B.7) to be negative are:

$$bx_2^{k-1} (k+1) > (1-b) (k-1) x_2 x_1^{k-2}$$

which amounts to

$$\frac{x_2}{x_1} < \left\{ \frac{b}{1-b} \frac{k+1}{k-1} \right\}^{\frac{1}{2-k}} \quad (\text{B.9})$$

And the other is:

$$2x_1^{k-1} (1-b) > (1-b) (k-1) x_1^{k-2} x_2$$

which amounts to:

$$\frac{x_2}{x_1} < \frac{2}{k-1} \quad (\text{B.10})$$

Recall that the equilibrium condition for the asymmetric costs is:

$$\frac{c_1}{c_2} = c = \frac{(1-b) x_1^{\alpha k-1} x_2^\alpha + b x_1^{\alpha-1} x_2^{\alpha k}}{(1-b) x_1^\alpha x_2^{\alpha k-1} + b x_1^{\alpha k} x_2^{\alpha-1}} \quad (\text{B.11})$$

Working on this condition we find the following:

$$x_1^{\alpha k-1} x_2^{\alpha-1} \{(1-b)x_2 - cbx_1\} = x_1^{\alpha-1} x_2^{\alpha k-1} \{c(1-b)x_1 - bx_2\} \quad (\text{B.12})$$

Note that for this equilibrium to hold the terms in the curly brackets cannot have opposite signs.

Therefore:

$$\frac{x_2}{x_1} \in \left(\min \left\{ \frac{c(1-b)}{b}, \frac{cb}{(1-b)} \right\}, \max \left\{ \frac{c(1-b)}{b}, \frac{cb}{(1-b)} \right\} \right) \quad (\text{B.13})$$

Now assume $b < \frac{1}{2}$ so by (B.13) $\frac{x_2}{x_1} < c \frac{(1-b)}{b}$. So by (B.9) and (B.10) the sufficient condition is satisfied if any of the following hold

$$\left(\frac{k+1}{k-1} \right)^{\frac{1}{2-k}} \left(\frac{b}{1-b} \right)^{\frac{k-1}{2-k}} > c \quad (\text{B.14})$$

$$\frac{2}{k-1} \frac{b}{1-b} > c \quad (\text{B.15})$$

It is clear that we can find parameters to satisfy the sufficient conditions. Frankly, as $k \rightarrow 2$, and $b \rightarrow 0$, those conditions are harder to fulfill. However with an example we show that even at the extremes we can have an equilibrium. So the sufficient conditions are much more restrictive than the necessary conditions.

Assume $\alpha = \frac{1}{2}$, $k = 2$, and $b = 0$. By (B.11), the equilibrium condition is $\frac{x_2}{x_1} = c^2$.

At this point the SOC of player one is, by (B.7):

$$x_2 - 2x_1 = x_1 (c^2 - 2) < 0 \Leftrightarrow 1 < c < \sqrt{2}$$

B.3 Estimation and test results

Table B.1: Estimation Results for three CSFs with home effect

n = 1381	Spanish League					
CSF	$\hat{\phi}$	$\hat{\alpha}$	\hat{k}	\hat{c}	\hat{a}	LL
ASF with $f(x) = x^\alpha$	1.540 (0.039)	0.486 (0.040)	1.426 (0.025)	-	-	-1347.8
ASF with $f(x) = \log(1+x)$	1.481 (0.069)	-	1.401 (0.023)	-	-	-1402.1
JSF with $f(x) = x^\alpha$	1.610 (0.045)	0.550 (0.045)	-	1.721 (0.050)	-	-1346.9
JSF with $f(x) = \log(1+x)$	1.553 (0.081)	-	-	1.641 (0.042)	-	-1405.8
BSF with $f(x) = x^\alpha$	1.790 (0.008)	0.563 (0.036)	-	-	89.5 (26.82)	-1356.9
BSF with $f(x) = \log(1+x)$	1.718 (0.112)	-	-	-	6.6 (0.5)	-1410.3
Naive	0.265	-	-	-	-	-1479.2
n = 1754	Italian League					
CSF	$\hat{\phi}$	$\hat{\alpha}$	\hat{k}	\hat{c}	\hat{a}	LL
ASF with $f(x) = x^\alpha$	1.599 (0.041)	0.560 (0.043)	1.544 (0.030)	-	-	-1731.9
ASF with $f(x) = \log(1+x)$	1.556 (0.072)	-	1.514 (0.027)	-	-	-1804.3
JSF with $f(x) = x^\alpha$	1.719 (0.048)	0.647 (0.049)	-	1.964 (0.066)	-	-1732.8
JSF with $f(x) = \log(1+x)$	1.664 (0.042)	-	-	1.852 (0.081)	-	-1809.7
BSF with $f(x) = x^\alpha$	1.965 (0.008)	0.675 (0.036)	-	-	329.8 (26.82)	-1749.7
BSF with $f(x) = \log(1+x)$	1.921 (0.116)	-	-	-	9.3 (0.6)	-1816.5
Naive	0.317	-	-	-	-	-1862.1
n = 1707	German League					
CSF	$\hat{\phi}$	$\hat{\alpha}$	\hat{k}	\hat{c}	\hat{a}	LL
ASF with $f(x) = x^\alpha$	1.425 (0.049)	0.611 (0.050)	1.482 (0.024)	-	-	-1733.1
ASF with $f(x) = \log(1+x)$	1.387 (0.062)	-	1.459 (0.025)	-	-	-1789.0
JSF with $f(x) = x^\alpha$	1.505 (0.058)	0.701 (0.057)	-	1.826 (0.048)	-	-1732.4
JSF with $f(x) = \log(1+x)$	1.454 (0.067)	-	-	1.750 (0.042)	-	-1792.4
BSF with $f(x) = x^\alpha$	1.664 (0.007)	0.734 (0.040)	-	-	402.8 (134.5)	-1741.5
BSF with $f(x) = \log(1+x)$	1.607 (0.095)	-	-	-	7.6 (0.5)	-1796.6
Naive	0.231	-	-	-	-	-1841.4
n = 1622	French League					
CSF	$\hat{\phi}$	$\hat{\alpha}$	\hat{k}	\hat{c}	\hat{a}	LL
ASF with $f(x) = x^\alpha$	1.539 (0.049)	0.393 (0.049)	1.562 (0.026)	-	-	-1684.7
ASF with $f(x) = \log(1+x)$	1.534 (0.062)	-	1.552 (0.025)	-	-	-1700.9
JSF with $f(x) = x^\alpha$	1.655 (0.058)	0.454 (0.058)	-	1.981 (0.055)	-	-1685.5
JSF with $f(x) = \log(1+x)$	1.653 (0.079)	-	-	1.947 (0.053)	-	-1703.3
BSF with $f(x) = x^\alpha$	1.947 (0.007)	0.444 (0.049)	-	-	45.1 (17.9)	-1694.4
BSF with $f(x) = \log(1+x)$	1.933 (0.122)	-	-	-	10.1 (0.7)	-1707.3
Naive	0.320	-	-	-	-	-1722.8

Note: This table reports the estimation results of the three CSF and the naive model. $\hat{\phi}$ is the home effect, $\hat{\alpha}$ is the decisiveness parameter. \hat{k} , \hat{c} , and \hat{a} correspond to model specific parameters of ASF, JSF and BSF, respectively. LL is the log-likelihood.

Table B.2: Estimation Results for three CSFs without home effect

n = 1381	Spanish League				
CSF	$\hat{\alpha}$	\hat{k}	\hat{c}	\hat{a}	LL
ASF with $f(x) = x^\alpha$	0.472 (0.037)	1.408 (0.023)	-	-	-1388.6
ASF with $f(x) = \log(1+x)$	-	1.386 (0.022)	-	-	-1440.3
JSF with $f(x) = x^\alpha$	0.532 (0.041)	-	1.677 (0.045)	-	-1388.0
JSF with $f(x) = \log(1+x)$	-	-	1.604 (0.039)	-	-
BSF with $f(x) = x^\alpha$	0.546 (0.044)	-	-	55.9 (21.01)	-1396.1
BSF with $f(x) = \log(1+x)$	-	-	-	4.9 (0.3)	-1447.9
Naive	-	-	-	-	-1517.2
n = 1754	Italian League				
CSF	$\hat{\alpha}$	\hat{k}	\hat{c}	\hat{a}	LL
ASF with $f(x) = x^\alpha$	0.538 (0.039)	1.512 (0.028)	-	-	-1801.4
ASF with $f(x) = \log(1+x)$	-	1.483 (0.026)	-	-	-1869.3
JSF with $f(x) = x^\alpha$	0.621 (0.043)	-	1.882 (0.049)	-	-1801.4
JSF with $f(x) = \log(1+x)$	-	-	1.783 (0.042)	-	-1874.5
BSF with $f(x) = x^\alpha$	0.641 (0.048)	-	-	167.9 (68.05)	-1816.9
BSF with $f(x) = \log(1+x)$	-	-	-	6.4 (0.3)	-1880.9
Naive	-	-	-	-	-1927.0
n = 1707	German League				
CSF	$\hat{\alpha}$	\hat{k}	\hat{c}	\hat{a}	LL
ASF with $f(x) = x^\alpha$	0.588 (0.047)	1.467 (0.022)	-	-	-1771.5
ASF with $f(x) = \log(1+x)$	-	1.447 (0.021)	-	-	-1823.5
JSF with $f(x) = x^\alpha$	0.672 (0.054)	-	1.785 (0.044)	-	-1771.3
JSF with $f(x) = \log(1+x)$	-	-	1.718 (0.040)	-	-1826.9
BSF with $f(x) = x^\alpha$	0.713 (0.061)	-	-	255.3 (128.9)	-1778.4
BSF with $f(x) = \log(1+x)$	-	-	-	5.8 (0.3)	-1830.6
Naive	-	-	-	-	-1875.3
n = 1622	French League				
CSF	$\hat{\alpha}$	\hat{k}	\hat{c}	\hat{a}	LL
ASF with $f(x) = x^\alpha$	0.391 (0.048)	1.533 (0.024)	-	-	-1743.1
ASF with $f(x) = \log(1+x)$	-	1.525 (0.024)	-	-	-1759.6
JSF with $f(x) = x^\alpha$	0.456 (0.055)	-	1.905 (0.050)	-	-1703.3
JSF with $f(x) = \log(1+x)$	-	-	1.872 (0.053)	-	-1761.8
BSF with $f(x) = x^\alpha$	0.441 (0.064)	-	-	29.9 (15.4)	-1752.6
BSF with $f(x) = \log(1+x)$	-	-	-	6.9 (0.7)	-1765.9
Naive	-	-	-	-	-1781.9

Note: This table reports the estimation results of the three CSF and the naive model without the home effect. $\hat{\alpha}$ is the decisiveness parameter. \hat{k} , \hat{c} , and \hat{a} correspond to model specific parameters of ASF, JSF and BSF, respectively. LL is the log-likelihood.

Table B.3: Vuong Test Results

Spanish League	Z-value	p-value
ASF vs. JSF	-1.069	0.285
ASF vs. BSF	1.486	0.137
JSF vs. BSF	1.625*	0.104
Italian League	Z-value	p-value
ASF vs. JSF	0.822	0.411
ASF vs. BSF	2.550***	0.011
JSF vs. BSF	2.389***	0.017
German League	Z-value	p-value
ASF vs. JSF	-0.851	0.395
ASF vs. BSF	1.527*	0.127
JSF vs. BSF	1.651**	0.099
French League	Z-value	p-value
ASF vs. JSF	0.847	0.397
ASF vs. BSF	1.521*	0.128
JSF vs. BSF	1.392	0.164

Note: This table reports the Vuong test results of the three CSFs. (***), (**), and (*) shows the one-tailed test significance levels of 1%, 5%, and 10%, respectively.

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